

# Relaxation of a test particle in systems with long-range interactions: diffusion coefficient and dynamical friction

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**Abstract.** We study the relaxation of a test particle immersed in a bath of field particles interacting via weak long-range forces. To order  $1/N$  in the  $N \rightarrow +\infty$  limit, the velocity distribution of the test particle satisfies a Fokker-Planck equation whose form is related to the Landau and Lenard-Balescu equations in plasma physics. We provide explicit expressions for the diffusion coefficient and friction force in the case where the velocity distribution of the field particles is isotropic. We consider (i) various dimensions of space  $d = 3, 2$  and  $1$ ; (ii) a discrete spectrum of masses among the particles; (iii) different distributions of the bath including the Maxwell distribution of statistical equilibrium (thermal bath) and the step function (water bag). Specific applications are given for self-gravitating systems in three dimensions, Coulombian systems in two dimensions and for the HMF model in one dimension.

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## 1 Introduction

Kinetic theories of many-particles systems are important to understand the dynamical evolution of the system and to determine transport properties. The first kinetic equation for a Hamiltonian  $N$ -body system was derived by Boltzmann in his theory of gases [1]. In that case, the particles do not interact except during strong collisions. Kinetic theories were later extended to the case of particles in interaction by Landau [2] in the case of plasmas and by Chandrasekhar [3] in the case of stellar systems. In this Introduction, we present a short historical review of kinetic equations with gravitational or coulombian interaction, stressing the literature before the sixties when these topics were developed. Additional references can be found in the books of Balescu [4] and Saslaw [5], or in the review of Kandrup [6]. The rest of the paper considers extensions of these kinetic theories to new situations.

Landau [2] derived his kinetic equation by starting from the Boltzmann equation and considering a weak deflection limit. Indeed, for a Coulombian potential of interaction slowly decreasing with the distance as  $r^{-1}$ , weak collisions are the most frequent ones. Each encounter induces a *small* change in the velocity of a particle but the cumulated effect of these encounters leads to a macroscopic process of diffusion in velocity space. This treatment, which assumes that the particles follow linear trajectories with constant velocity in a first approximation, yields a logarithmic divergence of the diffusion coefficient

for both small and large impact parameters but the equation can still be used successfully if appropriate cut-offs are introduced. A natural lower cut-off, which is called the Landau length, corresponds to the impact parameter leading to a deflection at  $90^\circ$ . On the other hand, in a neutral plasma, the potential is screened on a distance corresponding to the Debye length. Phenomenologically, the Debye length provides an upper cut-off. Later on, Lenard [7] and Balescu [8] developed a more precise (but also more formal) kinetic theory that could take into account collective effects. This gives rise to the inclusion of the dielectric function  $|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2$  in the denominator of the kinetic equation. Physically, this means that the particles are “dressed” by a polarization cloud. The original Landau equation, which ignores collective effects, is recovered from the Lenard-Balescu equation when  $|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2 = 1$ . However, with this additional term, it is found that the logarithmic divergence at large scales is now removed and that the Debye length is indeed the natural upper lengthscale to consider.

In stellar dynamics, Chandrasekhar [3] developed a kinetic theory in order to determine the timescale of collisional relaxation. He computed in particular the coefficients of diffusion and friction (second and first moments of the velocity increments) by considering the mean effect of a succession of two-body encounters. Since this approach can take into account large deflections, there is no divergence at small impact parameters and the gravitational analogue of the Landau length appears naturally. However, this approach leads to a logarithmic divergence at large scales that is more difficult to remove

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than in plasma physics because of the absence of Debye shielding for the gravitational force. In a series of papers, Chandrasekhar and von Neumann [9] developed a completely stochastic formalism of gravitational fluctuations and showed that the fluctuations of the gravitational force are given by the Holtzmark distribution (a particular Lévy law) in which the nearest neighbor plays a prevalent role. From these results, they argued that the logarithmic divergence has to be cut-off at the interparticle distance. However, since the interparticle distance is smaller than the Debye length, the same arguments should also apply in plasma physics, which is not the case. Therefore, the conclusions of Chandrasekhar and von Neumann [9] are usually taken with circumspection. It is usually argued (e.g., Cohen et al. [10], Saslaw [5], de Vega and Sanchez [11]) that the logarithmic divergence should be cut-off at the physical size  $R$  of the cluster for finite systems or at the Jeans scale for infinite systems, since the Jeans length is the presumable analogue of the Debye length in the present context (see Kandrup [6]). Chandrasekhar [12] also developed a Brownian theory of stellar dynamics and showed that, on a qualitative point of view, the results of kinetic theory could be understood very simply in that framework. In particular, he showed that the coefficients of diffusion and friction are related to each other by an Einstein relation [13]. Later on, Rosenbluth et al. [14] proposed a simplified derivation of the coefficients of diffusion and friction for plasmas and stellar systems and, substituting these expressions in the general form of the Fokker-Planck equation, they obtained a kinetic equation which is at the basis of the dynamics of stellar systems. They also provided simplified expressions of this equation in the case of axial symmetry.

It is interesting to note that the previous authors did not point out the link with the Landau equation and that modern textbooks of astrophysics [15] usually derive the kinetic equation of stellar dynamics from the Fokker-Planck equation by using the approach of Chandrasekhar [3] and Rosenbluth et al. [14]. We will see that we can equivalently obtain the kinetic equation of stellar dynamics from the Landau equation. This alternative derivation can re-inforce the connection between stellar systems and plasmas. We note, however, an important difference between stellar dynamics and plasma physics. Neutral plasmas are usually spatially homogeneous due to the Debye shielding. By contrast, stellar systems are inhomogeneous. Therefore, the above-mentioned kinetic theories developed in astrophysics rely on a *local approximation*. The collision term is calculated as if the system were homogeneous or as if the collisions could be treated as local. Then, the effect of inhomogeneity is taken into account in the kinetic equation by introducing an advective term (Vlasov term) in the left hand side which describes the evolution of the system due to mean-field effects. The local approximation is supported by the stochastic approach of Chandrasekhar and Von Neumann [9] showing the preponderance of the nearest neighbor. However, this remains a simplifying assumption which is not easily controllable. It is likely that the logarithmic di-

vergence at large scales comes from this approximation. More recently, Kandrup [16] derived a generalized Landau equation by using projection operator technics. This formal approach is interesting because it can take into account effects of spatial inhomogeneity and memory which are neglected in the previous approaches<sup>1</sup>. It clearly shows which approximations are needed in order to recover the Landau equation. However, the generalized Landau equation remains extremely complicated for practical purposes.

Until now, the kinetic theories of systems with long-range interactions have been essentially developed for 3D systems with Coulombian or Newtonian potential. The main object of this paper is to extend these theories to other dimensions of space  $d = 2$  and  $d = 1$  and for a wide class of potentials of interaction. This generalization has been initiated in Chavanis [21, 22]. It is shown that for systems with weak long-range interactions, the Landau and Lenard-Balescu equations describe the collisional dynamics of the system to order  $1/N$  in a proper thermodynamic limit  $N \rightarrow +\infty$ . Therefore, the collisional relaxation time scales in general as  $t_R \sim Nt_D$ , where  $t_D$  is the dynamical time. For  $N \rightarrow +\infty$  or  $t \ll t_R$ , the collision term is negligible and we obtain the (mean-field) Vlasov equation. In reference [22], we have introduced a Fokker-Planck equation which describes the evolution of a test particle in a bath of field particles. We have obtained analytical expressions of the diffusion coefficient and friction force in the case of a Maxwellian distribution of field particles (thermal bath). We shall obtain here generalizations of these results.

In the case of a Newtonian or Coulombian potential of interaction in  $d = 2$  the diffusion coefficient diverges *linearly* at large scales so that an upper cut-off (related to the Debye length in the case of a plasma) must be introduced. We have suggested in [22] that this divergence would be removed if we use the Lenard-Balescu kinetic equation taking into account collective effects as in the 3D case. This point will be further explored in the present paper. The kinetic theory of Coulombian interactions in  $d = 2$  has been studied independently by Benedetti et al. [23] using a different approach. They calculated the friction force experienced by a test particle by considering the effect of a succession of binary encounters. The calculations are difficult because they involve the differential cross section which has not an explicit form in  $d = 2$ . They however

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<sup>1</sup> Memory effects can be important for gravitating systems because the temporal correlation function of the force decreases algebraically as  $t^{-1}$  [17]. However, this slow decay mainly results in logarithmic divergences of the diffusion coefficient [18], so that the Landau equation can still be used successfully. Furthermore, as shown in Saslaw [5], non-Markovian effects are important for long-range collisions but not for short-range collisions. Since long-range collisions are more gentle than short-range collisions, memory effects are somehow ‘washed-out’ in the complete collision process. Memory effects can be important for other systems with long-range interactions, like the HMF model (see below), close to the critical point  $T_c$  because the timescale of the exponential decay of the correlation function diverges as  $T \rightarrow T_c$  [19–22].

managed to obtain asymptotic results for small and large velocities in the case where the distribution of the field particles is a step function. Our approach, based on the Landau or Lenard-Balescu equation in  $d = 2$ , does not require the expression of the differential cross section, just the Fourier transform of the potential of interaction. In this paper, we will calculate the diffusion coefficient and the friction force in the case where the distribution of the field particles is a step function and obtain results that are compatible and extend those of [23]. We will also discuss the stochastic properties of the fluctuating force created by a 2D Coulombian system by making a parallel with the approach of Chandrasekhar and von Neumann [9] for the gravitational force in  $d = 3$ .

In  $d = 1$ , the Landau and Lenard-Balescu collision terms cancel out indicating that the collisional evolution of the system as a whole is due to higher order correlations in the  $1/N$  expansion [22]. However, if we use this kinetic theory to describe the evolution of a test particle in a bath of field particles with any (stable) steady distribution of the Vlasov equation, we obtain a Fokker-Planck equation in which the diffusion coefficient and friction force can be easily calculated. This type of Fokker-Planck equations has been studied in [19,20,24,25] in connection with the HMF model. A kinetic theory of the HMF model has been developed in Bouchet [19] and Bouchet and Dauxois [24] by analyzing the stochastic process of fluctuations and calculating the first and second moments of the velocity increment  $\langle \Delta v \rangle$  and  $\langle (\Delta v)^2 \rangle$ . This is a particular case, for a cosine potential of interaction in one dimension, of the general Fokker-Planck approach developed in plasma physics (see Chap. 8 of Ichimaru [26]). These authors used this kinetic theory to determine the velocity correlation function and found an algebraic decay (see also [27,28]) which is consistent with direct numerical simulations of the  $N$ -body system. A kinetic theory of the HMF model was developed independently by Chavanis (see discussion in [20]) by using the projection operator formalism. This approach does not take into account collective effects but these terms can be obtained from the Lenard-Balescu theory [22]. Chavanis and Lemou [25] used this kinetic theory to study the relaxation of the distribution function tail and showed that it has a front structure and a slow (logarithmic) evolution. Non-ideal effects in the kinetic theory (non-Markovianity, spatial inhomogeneity,...) have been further discussed in [29,30]. In the present paper, we shall develop a kinetic theory of homogeneous one-dimensional systems with an arbitrary form of weak long-range potential of interaction, starting directly from the general Fokker-Planck equation given in [22], which takes into account collective effects. We shall determine the diffusion coefficient and analyze the effects of a mass distribution among the particles.

We note finally that kinetic theories of systems with long-range interactions have also been developed for point vortices in two-dimensional hydrodynamics and non-neutral plasmas confined by a magnetic field. Their form is related to, but different from, the Landau and Lenard-Balescu equations. The intrinsic reason is that point vor-

tices do not have inertia contrary to electric charges and stars. Hence, the coordinates  $x$  and  $y$  are canonically conjugate. Kinetic equations have been derived independently by Dubin and O'Neil [31,32] from the Klimontovich approach and by Chavanis [33] from projection operator technics. On the other hand, by using an analogy with stellar dynamics and Brownian theory, a Fokker-Planck equation describing the stochastic evolution of a test vortex in a bath of field vortices at equilibrium was derived in Chavanis [33,34]. The diffusion coefficient and the drift term are related to each other by an appropriate Einstein relation and they are inversely proportional to the local shear created by the field vortices. The statistics of the velocity fluctuations arising from a random distribution of point vortices has been investigated in Chavanis and Sire [35,36] by using an approach similar to that developed by Chandrasekhar and von Neumann [9] for the gravitational force. The analogy between stellar systems and 2D vortices is discussed in [37].

In this paper, we shall complete previous investigations on the kinetic theory of systems with weak long-range interactions by developing a general formalism valid for a large class of potentials of interaction in  $d$  dimensions and for multi-components systems. In Section 2.1, we recall the Lenard-Balescu and Landau equations describing the evolution of the distribution function (DF) of a system of particles in interaction. In Section 2.2, we introduce the Fokker-Planck equation describing the evolution of a test particle immersed in a bath of field particles. We give the general expressions of the coefficients of diffusion and friction. In Section 2.3, we restrict ourselves to isotropic distribution functions and derive the simplified form of the Fokker-Planck equation. We show how its stationary solutions are related to the distribution of the bath and we note that the test particle does not in general relax towards the distribution of the bath except (i) if it is Maxwellian (thermal bath); (ii) in  $d = 1$  for single-species particles. In Section 2.4, we consider the case where the field particles have a Maxwellian distribution. We find that the Fokker-Planck equation becomes similar to the Kramers equation but the diffusion is anisotropic and depends on the velocity of the test particle. The coefficients of diffusion and friction are related by an Einstein relation. We derive its general form for a multi-components system. In Section 2.5, we derive the explicit expression of the diffusion coefficient of a test particle in a thermal bath in  $d = 1, 2, 3$ . In Section 2.6, we estimate the relaxation time of the test particle to the Maxwellian distribution (thermalization) and show that it scales as  $Nt_D$ . We also investigate the effect of mass segregation on the relaxation time. Specific applications are given for gravitational systems and for the HMF model in Section 2.7. In Section 3, we consider the case  $d = 3$ . We show that the results obtained by Rosenbluth et al. [14] can be alternatively obtained from the Landau equation. This has not been noted by the previous authors and this can re-inforce the connection between plasma physics and stellar dynamics. As an application of the Rosenbluth potentials, we compute the diffusion coefficient and the friction force in the

case where the distribution of the bath is a step function. In Section 4, we consider the case  $d = 2$ . We obtain the general expressions of the diffusion coefficient and friction force valid for any isotropic distribution of the field particles in terms of integrals of Bessel functions. As an illustration, we compute them for the Coulombian interaction when the distribution of the bath is a step function and we obtain results similar to those of [23] up to a factor  $2(\pi - 2)/\pi$  that may be related to the unknown large scale cut-off. In Section 4.4, we analyze the statistics of fluctuations of the Coulombian force in  $d = 2$ . We show that it is given by a marginal Gaussian distribution intermediate between normal and Lévy laws. In particular, it presents an algebraic tail scaling as  $\sim F^{-4}$ . This is analogous to the distribution of the velocity created by a gas of point vortices in two-dimensions [35, 36]. We also discuss the origin of the linear divergence of the diffusion coefficient for Coulombian systems in  $d = 2$  and how this can be cured by considering collective effects. Finally, in Section 5 we provide the general form of the Fokker-Planck equation in  $d = 1$  and give explicit expressions of the diffusion coefficient and friction force for any (stable) steady distribution of the bath and in the case where the test particle has not necessarily the same mass as the field particles.

## 2 General results

### 2.1 Evolution of the system as a whole: the Landau and Lenard-Balescu equations

We consider a system of particles with long-range interactions whose dynamics is described by the Hamiltonian equations

$$m_i \frac{d\mathbf{r}_i}{dt} = \frac{\partial H}{\partial \mathbf{v}_i}, \quad m_i \frac{d\mathbf{v}_i}{dt} = -\frac{\partial H}{\partial \mathbf{r}_i}, \quad (1)$$

where

$$H = \sum_i^N m_i \frac{v_i^2}{2} + \sum_{i < j} \gamma_i \gamma_j u(\mathbf{r}_i - \mathbf{r}_j), \quad (2)$$

where  $u_{ij} = u(\mathbf{r}_i - \mathbf{r}_j)$  is a binary potential of interaction depending only on the absolute distance between particles. We assume that there exists  $X$  species of particles  $\{m_i, \gamma_i\}_{i=1, X}$  and we denote  $f_i(\mathbf{r}, \mathbf{v}, t)$  the distribution function of particles of species  $i$  normalized such that  $\int f_i d\mathbf{r} d\mathbf{v} = N_i m_i$  gives the total mass of particles of species  $i$ . We consider homogeneous systems and we assume that the potential of interaction is long-range and of weak amplitude. Then, the evolution of the distribution function of species  $i$  is given by the Lenard-Balescu equation

$$\frac{\partial f_i}{\partial t} = \pi(2\pi)^d \frac{\partial}{\partial v^\mu} \int d\mathbf{v}' d\mathbf{k} k^\mu k^\nu \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2} \delta(\mathbf{k} \cdot \mathbf{u}) \times \left( \frac{\gamma_i}{m_i} \right)^2 \sum_j \left( \frac{\gamma_j}{m_j} \right)^2 \left( m_j f_j' \frac{\partial f_i}{\partial v^\nu} - m_i f_i \frac{\partial f_j'}{\partial v'^\nu} \right) \quad (3)$$

with the dielectric function

$$\epsilon(\mathbf{k}, \omega) = 1 + (2\pi)^d \hat{u}(k) \sum_j \left( \frac{\gamma_j}{m_j} \right)^2 \int \frac{\mathbf{k} \cdot \frac{\partial f_j}{\partial \mathbf{v}}}{\omega - \mathbf{k} \cdot \mathbf{v}} d\mathbf{v}. \quad (4)$$

We have introduced the abbreviations  $f_i = f_i(\mathbf{v}, t)$  and  $f_i' = f_i(\mathbf{v}', t)$ . Furthermore,  $\mathbf{u} = \mathbf{v} - \mathbf{v}'$  is the relative velocity between two particles. It is also implicitly understood that there is summation over repeated Greek indices which denote the Cartesian coordinates of the vectors.

For systems with weak long-range interactions, the Lenard-Balescu equation gives the correction of order  $1/N$  to the Vlasov equation which is recovered for  $N \rightarrow +\infty$  (for homogeneous systems the Vlasov equation simply reduces to  $\partial f / \partial t = 0$ ). The proper thermodynamic limit is defined in [21, 22]. It is such that the amplitude of the potential of interaction scales as  $u \sim 1/N$  (weak coupling) while the energy per particle  $E/N$ , the inverse temperature  $\beta$  and the volume  $V$  are of the order unity. Noting that  $u^2 \sim 1/N^2$  and  $f \sim N$ , we find that the Lenard-Balescu collision term is of order  $1/N$ , i.e.  $\partial f / \partial t = \frac{1}{N} Q(f)$  with  $Q(f) \sim f$ . In  $d = 3$  and  $d = 2$ , it is easy to show that the Lenard-Balescu equation relaxes toward the Maxwellian

$$f_i^{eq} = A_i e^{-\beta m_i \frac{v_i^2}{2}}. \quad (5)$$

Therefore, finite  $N$  effects select the Maxwell distribution among all possible stationary solutions of the Vlasov equation. The collisional relaxation time scales as  $t_R \sim N t_D$  where  $t_D$  is a dynamical time (which can be taken of order unity). For Newtonian interactions in  $d = 3$ , there is a logarithmic correction  $\ln A \sim \ln N$  so that the collisional relaxation time scales as  $t_R \sim (N / \ln N) t_D$ . Alternatively, for  $d = 1$ , the Lenard-Balescu operator cancels out so that the collisional evolution is due to terms of higher order in  $1/N$ . This implies that the collisional relaxation time of the system as a whole scales as  $t_R \sim N^\delta t_D$  with  $\delta > 1$ .

If we neglect collective effects and take  $|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2 = 1$  we obtain the Landau equation

$$\frac{\partial f_i}{\partial t} = \pi(2\pi)^d \frac{\partial}{\partial v^\mu} \int d\mathbf{v}' d\mathbf{k} k^\mu k^\nu \hat{u}(k)^2 \delta(\mathbf{k} \cdot \mathbf{u}) \times \left( \frac{\gamma_i}{m_i} \right)^2 \sum_j \left( \frac{\gamma_j}{m_j} \right)^2 \left( m_j f_j' \frac{\partial f_i}{\partial v^\nu} - m_i f_i \frac{\partial f_j'}{\partial v'^\nu} \right) \quad (6)$$

as an approximation of the Lenard-Balescu equation. An important remark for the following is that the expression (6) of the Landau equation does not involve the differential cross section of the interaction, but only the Fourier transform of the potential<sup>2</sup>. Furthermore, as discussed after equation (73), the potential of interaction only fixes the

<sup>2</sup> The Fourier transform of the potential is just the Born approximation to the scattering amplitude. Therefore, the assumptions that are made to obtain the Landau equation amount to a first order perturbative approximation for the elastic two-body differential cross section (I thank one of the referees for this remark).

timescale of relaxation so that the *structure* of the Landau equation does not depend on the potential. This is a consequence of the fact that the amplitude of the interaction is very small so that a weak deflection approximation is appropriate and yields results that are relatively independent on the precise form of the potential of interaction.

The Landau equation (6) can be derived from the Boltzmann [2] or from the Fokker-Planck equation [3,14] by using a model of binary collisions. The Lenard-Balescu equation (3) can be obtained from the Liouville equation by using iterative procedures and diagrammatic methods [8]. It can also be obtained from the BBGKY hierarchy [26] or from the Klimontovich equation [38] by using approximations which amount to neglecting some correlations. This can be justified perturbatively when we consider an expansion of the equations of the problem in terms of the small parameter  $\sim 1/N$  (weak coupling limit). These derivations remain valid for other types of potentials with weak amplitude and other dimensions of space so that equations (3) and (6) have a larger domain of validity than plasma physics and stellar dynamics. These equations will be the starting point of our analysis.

## 2.2 Evolution of a test particle in a bath: the Fokker-Planck equation

We now consider the evolution of a test particle ( $m, \gamma$ ) in a bath of field particles with prescribed distribution  $f_j(\mathbf{v})$  for each species ( $m_j, \gamma_j$ ). It has been established in plasma physics (see, e.g. [26]) that the evolution of the velocity distribution  $P(\mathbf{v}, t)$  of the test particle is governed by a Fokker-Planck equation that takes a form similar to the Lenard-Balescu equation (3) provided that we replace  $f_j(\mathbf{v}', t)$  by the *prescribed* distribution  $f_j(\mathbf{v}')$  of the bath. This general Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \pi(2\pi)^d \frac{\partial}{\partial v^\mu} \int d\mathbf{v}' d\mathbf{k} k^\mu k^\nu \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2} \delta(\mathbf{k} \cdot \mathbf{u}) \times \left(\frac{\gamma}{m}\right)^2 \sum_j \left(\frac{\gamma_j}{m_j}\right)^2 \left(m_j f'_j \frac{\partial P}{\partial v^\nu} - m P \frac{\partial f'_j}{\partial v'^\nu}\right), \quad (7)$$

will be our main object of interest. In the following, in order to simplify the expressions, we shall take  $\gamma_i = m_i$  (like for the gravitational interaction) but the generalization of our results for  $\gamma_i \neq m_i$  is straightforward. Furthermore, for  $d = 3$  and  $d = 2$  we shall make the Landau approximation  $|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2 = 1$  (some results that relax the Landau approximation are given in [22] for a thermal bath). In that case, the Fokker-Planck equation for the test particle becomes

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v^\mu} \sum_j \int K^{\mu\nu} \left(m_j f'_j \frac{\partial P}{\partial v^\nu} - m P \frac{\partial f'_j}{\partial v'^\nu}\right) d\mathbf{v}' \quad (8)$$

where

$$K^{\mu\nu} = \pi(2\pi)^d \int k^\mu k^\nu \hat{u}(k)^2 \delta(\mathbf{k} \cdot \mathbf{u}) d\mathbf{k}. \quad (9)$$

We introduce the diffusion tensor and the friction term:

$$D^{\mu\nu} = \sum_j \int K^{\mu\nu} m_j f'_j d\mathbf{v}' \equiv \sum_j D^{\mu\nu}(j \rightarrow 0), \quad (10)$$

$$\eta^\mu = -m \sum_j \int K^{\mu\nu} \frac{\partial f'_j}{\partial v'^\nu} d\mathbf{v}' \equiv \sum_j \eta^\mu(j \rightarrow 0), \quad (11)$$

where  $D^{\mu\nu}(j \rightarrow 0)$  and  $\eta^\mu(j \rightarrow 0)$  are the diffusion and the friction caused by species  $j$  on the test particle denoted 0. With these notations, equation (8) can be rewritten

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v^\mu} \left[ D^{\mu\nu} \frac{\partial P}{\partial v^\nu} + P \eta^\mu \right]. \quad (12)$$

Comparing with the general expression of the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial v^\mu \partial v^\nu} \left( \frac{\langle \Delta v^\mu \Delta v^\nu \rangle}{\Delta t} P \right) + \frac{\partial}{\partial v^\mu} \left( \frac{\langle \Delta v^\mu \rangle}{\Delta t} P \right), \quad (13)$$

we find that

$$\frac{\langle \Delta v^\mu \Delta v^\nu \rangle}{\Delta t} = 2D^{\mu\nu} \quad (14)$$

$$\frac{\langle \Delta v^\mu \rangle}{\Delta t} = \eta^\mu - \frac{\partial D^{\mu\nu}}{\partial v^\nu}. \quad (15)$$

Now, using the fact that  $K^{\mu\nu}$  depends only on the relative velocity  $\mathbf{u} = \mathbf{v} - \mathbf{v}'$  and using an integration by parts, we get

$$\frac{\partial D^{\mu\nu}}{\partial v^\nu} = \sum_j \int \frac{\partial K^{\mu\nu}}{\partial v^\nu} m_j f'_j d\mathbf{v}' = - \sum_j \int \frac{\partial K^{\mu\nu}}{\partial v'^\nu} m_j f'_j d\mathbf{v}' = \sum_j \int K^{\mu\nu} m_j \frac{\partial f'_j}{\partial v'^\nu} d\mathbf{v}'. \quad (16)$$

Therefore, the first moment of the velocity increment can be rewritten

$$\begin{aligned} \frac{\langle \Delta v^\mu \rangle}{\Delta t} &= - \sum_j \int K^{\mu\nu} (m + m_j) \frac{\partial f'_j}{\partial v'^\nu} d\mathbf{v}' \\ &= - \sum_j \frac{m + m_j}{m} \eta^\mu(j \rightarrow 0). \end{aligned} \quad (17)$$

We note that because of the velocity dependence of the diffusion tensor, the friction terms  $\langle \Delta \mathbf{v} \rangle / \Delta t$  and  $\boldsymbol{\eta}$  do not coincide. In particular, for equal mass particles, there is a factor 2 between them as noted previously [39]. On the other hand, if the field particles have a mass  $m_f$  different from the mass  $m$  of the test particle, the multiplicative factor is  $(m + m_f)/m$ . In this respect, we note that the frictional force  $\langle \mathbf{F}_{\text{fr}} \rangle$  calculated by Kandrup [40] with his linear response theory is  $\boldsymbol{\eta}$ , not  $\langle \Delta \mathbf{v} \rangle / \Delta t$ . This explains

why it differs from the calculation of Chandrasekhar [3] by a factor  $(m + m_f)/m$  (see [40], pp. 446). The frictional force  $\boldsymbol{\eta}$  is an important quantity by itself because it is the quantity which naturally appears in the *symmetrical* form of the Landau-Lenard-Balescu collision term where the diffusion coefficient is placed between the two derivatives:  $\partial_\mu D^{\mu\nu} \partial_\nu$ , see equations (3), (6) and (12).

### 2.3 The isotropic Fokker-Planck equation

If the distribution function of the field particles is isotropic, i.e.  $f_j(\mathbf{v}) = f_j(v)$ , we can write the diffusion tensor in the form

$$D^{\mu\nu} = \left( D_{\parallel} - \frac{1}{d-1} D_{\perp} \right) \frac{v^\mu v^\nu}{v^2} + \frac{1}{d-1} D_{\perp} \delta^{\mu\nu}, \quad (18)$$

where  $D_{\parallel}(v)$  and  $D_{\perp}(v)$  are the diffusion coefficients in the directions parallel and perpendicular to the direction of the test particle  $\mathbf{v}$ . On the other hand, starting from equations (10) and (11) and using the same type of calculation as in equation (16), we find that the friction term can be generally written

$$\eta^\mu = -m \sum_j \frac{1}{m_j} \frac{\partial}{\partial v^\nu} D^{\mu\nu} (j \rightarrow 0). \quad (19)$$

For an isotropic distribution of the field particles, using equation (18), we find that the friction term is parallel to the velocity of the test particle, i.e.  $\boldsymbol{\eta} = \eta \mathbf{v}/v$ . The amplitude of the friction vector is given by

$$\eta = -m \sum_j \frac{1}{m_j} \left[ \frac{dD_{\parallel}}{dv} + \frac{d-1}{v} \left( D_{\parallel} - \frac{D_{\perp}}{d-1} \right) \right] (j \rightarrow 0). \quad (20)$$

If the velocity distribution of the test particle is itself isotropic, i.e.  $P(\mathbf{v}, t) = P(v, t)$ , we can rewrite the Fokker-Planck equation (12) in the form

$$\frac{\partial P}{\partial t} = \frac{1}{v^{d-1}} \frac{\partial}{\partial v} \left[ v^{d-1} \left( D_{\parallel} \frac{\partial P}{\partial v} + P \eta \right) \right] \quad (21)$$

where we have used

$$D^{\mu\nu} v^\nu = \left( D_{\parallel} - \frac{1}{d-1} D_{\perp} \right) v^\mu + \frac{1}{d-1} D_{\perp} v^\mu = D_{\parallel} v^\mu. \quad (22)$$

Equation (21) can also be written as

$$\frac{\partial P}{\partial t} = \frac{1}{v^{d-1}} \frac{\partial}{\partial v} \left[ v^{d-1} D_{\parallel}(v) \left( \frac{\partial P}{\partial v} + P \frac{dU}{dv} \right) \right] \quad (23)$$

where we have introduced the effective potential

$$U(v) = \int^v \frac{\eta(v')}{D_{\parallel}(v')} dv'. \quad (24)$$

Equation (23) relaxes towards a stationary solution of the form

$$P^{eq}(v) = A e^{-U(v)}, \quad (25)$$

provided that this distribution is normalizable. The evolution of the high velocity tail of the distribution function that is solution of a Fokker-Planck equation of the general form (23) has been studied in [25].

Consider single species systems. For  $d = 2, 3$ , the *only* stationary solution of the Lenard-Balescu equation (3) is the Maxwellian (5). This implies that the stationary solution of the Fokker-Planck equation (7) will be equal to the distribution of the bath  $f(\mathbf{v})$  *only* if this distribution is the Maxwellian. Otherwise,  $P^{eq}(v)$  is not equal to the distribution  $f(\mathbf{v})$  of the bath. We shall give an explicit example in Section 3.3. By contrast, for  $d = 1$ , the Lenard-Balescu operator cancels out for *any* distribution. This implies that the stationary solution of the Fokker-Planck equation (7) is *always* equal to the distribution of the bath  $f(\mathbf{v})$ , even if it is not the Maxwellian (see Sect. 5).

### 2.4 The Einstein relation for isothermal systems

If the velocity distribution of the field particles is the Maxwellian (thermal bath)

$$f_j = A_j e^{-\beta m_j \frac{v^2}{2}}, \quad (26)$$

then

$$\frac{\partial f'_j}{\partial v'^\nu} = -\beta m_j f'_j v'^\nu. \quad (27)$$

Substituting this relation in equation (11) and using the relation

$$K^{\mu\nu} v'^\nu = K^{\mu\nu} (v^\nu - u^\nu) = K^{\mu\nu} v^\nu, \quad (28)$$

which results from the identity  $K^{\mu\nu} u^\nu = 0$  ( $K^{\mu\nu}$  is the projector in the direction perpendicular to  $\mathbf{u}$ ), we get

$$\eta^\mu = \beta m D^{\mu\nu} v^\nu. \quad (29)$$

This can be viewed as the general expression of the Einstein relation in our context (this is the most general relation that equation (12) must satisfy in order to admit the Maxwell distribution as a stationary state). In that case, the Fokker-Planck equation (12) can be written in a form similar to the Kramers equation [41], but with an anisotropic tensor depending on the velocity of the test particle:

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v^\mu} \left[ D^{\mu\nu}(v) \left( \frac{\partial P}{\partial v^\nu} + \beta m P v^\nu \right) \right]. \quad (30)$$

The stationary solution of this equation is the Maxwellian

$$P^{eq}(v) = A e^{-\beta m \frac{v^2}{2}}. \quad (31)$$

Note also that according to equation (22) we have

$$\boldsymbol{\eta} = \beta m D_{\parallel} \mathbf{v}. \quad (32)$$

Therefore, if we assume that the velocity distribution of the test particle is isotropic we get

$$\frac{\partial P}{\partial t} = \frac{1}{v^{d-1}} \frac{\partial}{\partial v} \left[ v^{d-1} D_{\parallel}(v) \left( \frac{\partial P}{\partial v} + \beta m P v \right) \right]. \quad (33)$$

If we momentarily neglect the velocity dependence of the diffusion coefficient, this Fokker-Planck equation can be obtained from a Langevin stochastic process of the form

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \quad \frac{d\mathbf{v}}{dt} = -\xi \mathbf{v} + \sqrt{2D_{\parallel}} \mathbf{R}(t), \quad (34)$$

where  $\mathbf{R}(t)$  is a white noise satisfying  $\langle \mathbf{R}(t) \rangle = 0$  and  $\langle R_i(t) R_j(t') \rangle = \delta_{ij} \delta(t - t')$  and the friction coefficient is given by the Einstein relation  $\xi = \beta m D_{\parallel}$ . This relation is *necessary* to obtain the Maxwellian distribution at equilibrium and this is why it is rather independent on the details of the microscopic model.

As indicated previously, since the diffusion coefficient depends on the velocity,  $\boldsymbol{\eta}$  is not exactly the friction force so that equation (32) is not the proper form of Einstein relation. Using equation (17) we have

$$\frac{\langle \Delta \mathbf{v} \rangle}{\Delta t} = -\beta \sum_j (m + m_j) D_{\parallel}(j \rightarrow 0) \mathbf{v}. \quad (35)$$

If the field particles all have the same mass  $m_f$  we obtain

$$\frac{\langle \Delta \mathbf{v} \rangle}{\Delta t} = -\beta (m + m_f) D_{\parallel} \mathbf{v}, \quad (36)$$

so that the proper friction coefficient is

$$\xi' = \beta (m + m_f) D_{\parallel}. \quad (37)$$

This is the proper form of the Einstein relation in the present context. Note that it involves the sum of the mass of the test particle and of the field particles. Using

$$\frac{1}{2} m_f \langle v^2 \rangle_f = \frac{d}{2} k_B T = \frac{d}{2\beta}, \quad (38)$$

the Einstein relation (37) can be rewritten

$$\frac{\langle (\Delta v_{\parallel})^2 \rangle}{\xi' \Delta t} = \frac{2}{d} \frac{m_f}{m + m_f} \langle v^2 \rangle_f. \quad (39)$$

## 2.5 The diffusion coefficient for isothermal systems

Inserting the identity

$$\delta(x) = \int_{-\infty}^{+\infty} e^{itx} \frac{dt}{2\pi}, \quad (40)$$

in equation (9) and performing the Fourier transform on  $\mathbf{v}'$  of the Gaussian distribution (26) in equation (10), we

find after another Fourier transform on  $t$  that the diffusion tensor can be expressed as

$$D^{\mu\nu} = \pi (2\pi)^d \left( \frac{\beta}{2\pi} \right)^{1/2} \sum_j \rho_j m_j^{3/2} \times \int d\mathbf{k} \frac{k^{\mu} k^{\nu}}{k} \hat{u}(k)^2 e^{-\beta m_j \frac{(\mathbf{k} \cdot \mathbf{v})^2}{2k^2}}. \quad (41)$$

This can be rewritten

$$D^{\mu\nu} = \pi (2\pi)^d \left( \frac{\beta}{2\pi} \right)^{1/2} \sum_j \rho_j m_j^{3/2} \times \int_0^{+\infty} k^d \hat{u}(k)^2 dk G^{\mu\nu} \left( \sqrt{\frac{\beta m_j}{2}} v \right), \quad (42)$$

where

$$G^{\mu\nu}(x) = \int d\hat{\mathbf{k}} \hat{k}^{\mu} \hat{k}^{\nu} e^{-(\hat{\mathbf{k}} \cdot \mathbf{x})^2}, \quad (43)$$

where we have noted  $\hat{\mathbf{k}} = \mathbf{k}/k$ . In  $d = 3$ , introducing a spherical system of coordinates with the  $z$ -axis in the direction of  $\mathbf{x}$ , we obtain

$$G^{\mu\nu} = \left( G_{\parallel} - \frac{1}{2} G_{\perp} \right) \frac{x^{\mu} x^{\nu}}{x^2} + \frac{1}{2} G_{\perp} \delta^{\mu\nu}, \quad (44)$$

with

$$G_{\parallel} = \frac{2\pi^{3/2}}{x} G(x), \quad G_{\perp} = \frac{2\pi^{3/2}}{x} [\Phi(x) - G(x)], \quad (45)$$

where

$$G(x) = \frac{2}{\sqrt{\pi}} \frac{1}{x^2} \int_0^x t^2 e^{-t^2} dt = \frac{\Phi(x) - x\Phi'(x)}{2x^2}, \quad (46)$$

and  $\Phi(x)$  is the error function

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \quad (47)$$

In  $d = 2$ , using a polar system of coordinates with the  $x$ -axis in the direction of  $\mathbf{x}$ , we get

$$G^{\mu\nu} = (G_{\parallel} - G_{\perp}) \frac{x^{\mu} x^{\nu}}{x^2} + G_{\perp} \delta^{\mu\nu}, \quad (48)$$

with

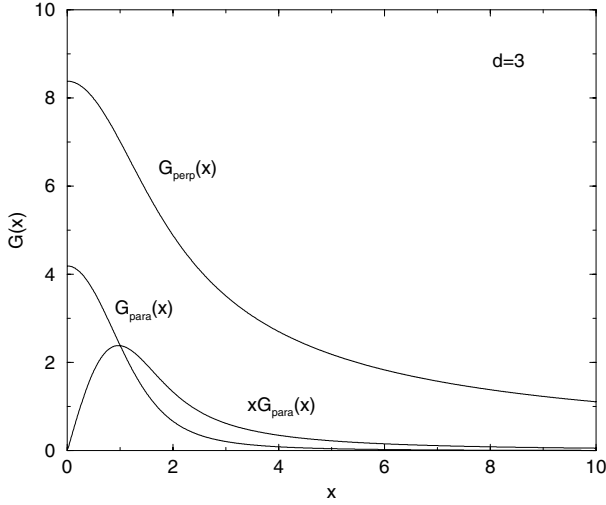
$$G_{\parallel} = \pi e^{-\frac{x^2}{2}} \left[ I_0 \left( \frac{x^2}{2} \right) - I_1 \left( \frac{x^2}{2} \right) \right], \quad (49)$$

$$G_{\perp} = \pi e^{-\frac{x^2}{2}} \left[ I_0 \left( \frac{x^2}{2} \right) + I_1 \left( \frac{x^2}{2} \right) \right]. \quad (50)$$

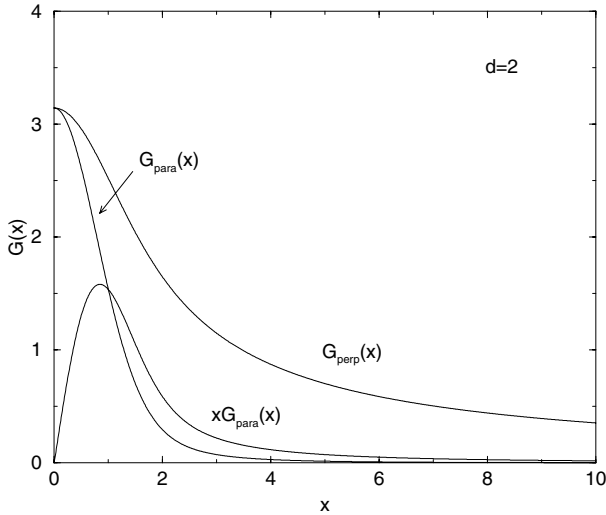
Finally, in  $d = 1$ , we obtain

$$G(x) = 2e^{-x^2}. \quad (51)$$

The diffusion coefficients and friction force are plotted in Figures 1–3 for different dimensions of space.



**Fig. 1.** Diffusion coefficients  $G_{\parallel}(x)$ ,  $G_{\perp}(x)$  and friction force  $xG_{\parallel}(x)$  for a thermal bath in  $d = 3$ .



**Fig. 2.** Diffusion coefficients  $G_{\parallel}(x)$ ,  $G_{\perp}(x)$  and friction force  $xG_{\parallel}(x)$  for a thermal bath in  $d = 2$ .

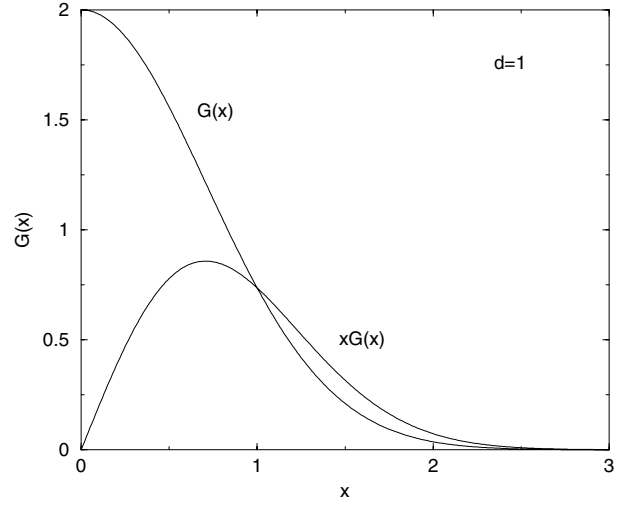
## 2.6 The relaxation time

We can use the above results to estimate the relaxation time of the test particle towards the Maxwellian distribution (thermalization). We consider the relaxation of a test particle with mass  $m$  in a thermal bath of field particles with mass  $m_f$ . If we set  $\mathbf{x} = \sqrt{\beta m_f/2} \mathbf{v}$ , the Fokker-Planck equation (30) can be rewritten

$$\frac{\partial P}{\partial t} = \frac{1}{t_R} \frac{\partial}{\partial x^\mu} \left[ G^{\mu\nu}(x) \left( \frac{\partial P}{\partial x^\nu} + 2 \frac{m}{m_f} P x^\nu \right) \right], \quad (52)$$

where  $t_R$  is a reference time given by

$$\frac{1}{t_R} = \left( \frac{\pi}{8} \right)^{1/2} d^{3/2} (2\pi)^d \frac{\rho m_f}{v_{mf}^3} \int_0^{+\infty} k^d \hat{u}(k)^2 dk, \quad (53)$$



**Fig. 3.** Diffusion coefficient  $G(x)$  and friction force  $xG(x)$  for a thermal bath in  $d = 1$ .

where  $v_{mf}^2 = d/(\beta m_f)$  is the mean-squared velocity of the field particles. Introducing the dimensionless function

$$\eta(k) = -(2\pi)^d \hat{u}(k) \beta m_f \rho, \quad (54)$$

we can rewrite the reference time (53) in the form

$$\frac{1}{t_R} = \left( \frac{\pi}{8d} \right)^{1/2} \frac{v_{mf}}{nL^{d+1}} \frac{1}{(2\pi)^d} \int_0^{+\infty} \kappa^d \eta(\kappa/L)^2 d\kappa, \quad (55)$$

where  $L = V^{1/d}$  is the size of the system and  $\kappa = kL$  is dimensionless ( $n = \rho/m_f$  is the numerical density of the field particles). We define a typical dynamical time by

$$t_D = \frac{L}{v_{mf}}. \quad (56)$$

Then, introducing the number  $N = nL^d$  of field particles, we can finally write the reference time in the form

$$t_R = C_d N t_D, \quad (57)$$

where

$$C_d^{-1} = \left( \frac{\pi}{8d} \right)^{1/2} \frac{1}{(2\pi)^d} \int_0^{+\infty} \kappa^d \eta(\kappa/L)^2 d\kappa, \quad (58)$$

is a dimensionless number.

We can get an estimate of the relaxation time by the following argument. If the diffusion coefficient were constant, the typical velocity of the test particle (in one spatial direction) would increase like

$$\frac{1}{d} \langle (\Delta \mathbf{v})^2 \rangle \sim 2D_{\parallel} t. \quad (59)$$

The relaxation time  $t_r$  is the typical time at which the typical velocity of the test particle has reached its equilibrium value  $\langle v^2 \rangle(+\infty) = d/(m\beta) = (m_f/m)v_{mf}^2$  so that



$\langle(\Delta\mathbf{v})^2\rangle(t_r) = \langle v^2\rangle(+\infty)$ . Since  $D_{\parallel}$  depends on  $v$ , the description of the diffusion is more complex. However, the formula

$$t_r = \frac{1}{d} \frac{m_f}{m} \frac{v_{mf}^2}{2D_{\parallel}(v_{mf})}, \quad (60)$$

resulting from the previous arguments with  $D_{\parallel} = D_{\parallel}(v_{mf})$  should provide a good estimate of the relaxation time. Using equation (42) and comparing with equation (55) we obtain

$$t_r = K_d \frac{m_f}{m} t_R, \quad (61)$$

where  $K_d = 1/[4G_{\parallel}(\sqrt{d/2})]$ . We find  $K_3 = 0.13587547\dots$ ,  $K_2 = 0.16286327\dots$  and  $K_1 = 0.20609016\dots$ . We can also estimate the relaxation time by  $t'_r = \xi^{-1}$  where  $\xi$  is the friction coefficient. Using the Einstein relation  $\xi = D_{\parallel}\beta m$  with  $D_{\parallel} = D_{\parallel}(v_{mf})$  we find that

$$t'_r = 2t_r. \quad (62)$$

Combining the previous results we find that the relaxation time scales as

$$t_r \sim N \frac{m_f}{m} t_D. \quad (63)$$

We expect that this result also yields a good estimate of the relaxation time of a test particle towards the stationary distribution (25) in the case of a non-thermal bath. In that case, we must justify that the DF of the bath does not change on that interval (see below). We come therefore to the following conclusions. For equal mass particles, the relaxation time of a test particle in a bath scales as  $Nt_D$ . For  $d = 2, 3$ , the relaxation time of the system as a whole also scales as  $Nt_D$  (see Sect. 2.1). Therefore, a non-thermal bath will change on this timescale. Only the Maxwellian distribution is stationary on the timescale  $Nt_D$ . Thus, for  $m = m_f$ , the test particle approach can be developed only for a thermal bath. Now, consider the relaxation of a test particle with mass  $m$  in a bath of field particles with mass  $m_f$ . In that case, the relaxation of the test particle is changed by a factor  $m_f/m$ . If  $m \gg m_f$  we find that the relaxation of the test particle ( $\sim(m_f/m)Nt_D$ ) towards equation (25) is much faster than the relaxation of the system of field particles as a whole ( $\sim Nt_D$ ) towards the Maxwellian  $f^{eq}$ . Therefore, in that limit, when we focus on the evolution of a test particle, it is possible to consider that the distribution of the field particles  $f(\mathbf{v}, t)$  is ‘‘frozen’’ even if it does not correspond to statistical equilibrium. This is because  $f(\mathbf{v}, t)$  evolves much slower than  $P(\mathbf{v}, t)$ . This remark justifies to consider equations of the form (7) with  $f \neq f^{eq}$ . In astrophysics, the case  $m \gg m_f$  could be relevant to describe the stochastic evolution of a black hole at the center of a galaxy or the dynamics of a globular cluster ( $m \sim 10^6 m_f$ ) passing through a galaxy. Note that in  $d = 1$  the relaxation of a test particle in a bath is given by equation (63) while the relaxation of the field particles as a whole towards statistical equilibrium is *larger* than  $Nt_D$  since the Lenard-Balescu collision term

cancels out in  $d = 1$ . Therefore, in that case it is justified to consider the relaxation of a test particle in a bath with *any* distribution  $f(v)$  (stable with respect to the Vlasov equation) even if  $m = m_f$  (see Sect. 5).

## 2.7 Examples

Let us apply these results to particular systems considered in [21, 22]. For the gravitational interaction in  $d = 3$ , the Fourier transform of the potential is

$$(2\pi)^3 \hat{u}(k) = -\frac{4\pi G}{k^2}. \quad (64)$$

The reference time (53) can be written

$$t_R = 0.482 \frac{v_{mf}^3}{nm_f^2 G^2 \ln \Lambda}, \quad (65)$$

where  $\ln \Lambda$  is the Coulombian factor [15]. Using equations (54) and (64), we get

$$\eta(k) = \frac{k_J^2}{k^2}, \quad (66)$$

where  $k_J = (4\pi G\beta\rho m_f)^{1/2}$  is the Jeans wavenumber. Then, equations (57–58) lead to

$$t_R = \frac{11.3}{\eta^2} \frac{N}{\ln \Lambda} t_D, \quad (67)$$

where we have defined  $\eta = \beta GNm_f^2/R$  with  $L^3 = V = (4/3)\pi R^3$  and we recall that  $\ln \Lambda \sim \ln N$ .

For the HMF model in  $d = 1$ , we have

$$\hat{u}_n = -\frac{k}{4\pi}(\delta_{n,1} + \delta_{n,-1}). \quad (68)$$

The reference time (53) can be written

$$t_R = 40.1 \frac{v_{mf}^3}{nm_f^2 k^2}, \quad (69)$$

where we have replaced  $\int_0^{+\infty} dk$  by  $\sum_{n=0}^{+\infty}$  in the discrete case. Using equations (54) and (68), we get

$$\eta_n = \eta(\delta_{n,1} + \delta_{n,-1}), \quad \eta = \frac{\beta Nm_f^2 k}{4\pi}. \quad (70)$$

Then, we obtain

$$t_R = \frac{0.254}{\eta^2} Nt_D. \quad (71)$$

For the HMF model, the relaxation towards the Maxwellian is not exponential due to the rapid decay of the diffusion coefficient [24, 25] so that this reference time only gives a timescale of relaxation.

### 3 The case $d = 3$

#### 3.1 Rosenbluth potentials

In  $d = 3$ , it is possible to obtain a simple expression of the coefficients of diffusion and friction, expressed in terms of elementary integrals, for *any* isotropic distribution of the bath (not only the Maxwellian). To that purpose, it is useful to introduce the so-called Rosenbluth potentials [14, 15, 39]. Introducing a spherical system of coordinates with the  $z$ -axis in the direction of  $\mathbf{u}$ , we find after elementary calculations that equation (9) can be written

$$K^{\mu\nu} = A_3 \frac{\delta^{\mu\nu} u^2 - u^\mu u^\nu}{u^3} \quad (72)$$

where

$$A_3 = 8\pi^5 \int_0^{+\infty} k^3 \hat{u}(k)^2 dk. \quad (73)$$

In particular, for the gravitational potential we find that  $A_3 = 2\pi G^2 \ln \Lambda$  where  $\ln \Lambda = \int_0^{+\infty} dk/k$  is the Coulombian factor which has to be regularized with appropriate cut-offs (see the Introduction). Using equation (72), the Landau equation (6) can be written for a single species system

$$\frac{\partial f}{\partial t} = A \frac{\partial}{\partial v^\mu} \int \frac{\delta^{\mu\nu} u^2 - u^\mu u^\nu}{u^3} \left( f' \frac{\partial f}{\partial v^\nu} - f \frac{\partial f'}{\partial v^\mu} \right) d\mathbf{v}' \quad (74)$$

with  $A = 2\pi G^2 m \ln \Lambda$ . Equation (74) is the original form of the Landau equation (we have used the notations of astrophysics but the connection with the notations of plasma physics is obtained by making the substitution  $Gm^2 \leftrightarrow -e^2$ ). It applies to weakly coupled systems. We note that the potential of interaction appears only in the constant  $A$  which merely determines the timescale of relaxation. The structure of the Landau equation is independent on the potential. Using the identity

$$\frac{\partial^2 u}{\partial v^\mu \partial v^\nu} = \frac{\delta^{\mu\nu} u^2 - u^\mu u^\nu}{u^3}, \quad (75)$$

we can write the diffusion tensor (10) in the form

$$D^{\mu\nu} = A_3 \sum_j m_j \frac{\partial^2 g_j}{\partial v^\mu \partial v^\nu}(\mathbf{v}), \quad (76)$$

where

$$g(\mathbf{v}) = \int f(\mathbf{v}') |\mathbf{v} - \mathbf{v}'| d\mathbf{v}'. \quad (77)$$

On the other hand, integrating by parts and using the fact that  $K^{\mu\nu}$  depends only on the relative velocity  $\mathbf{u} = \mathbf{v} - \mathbf{v}'$ , the friction term (11) can be written

$$\eta^\mu = -m \sum_j \int \frac{\partial K^{\mu\nu}}{\partial v^\nu} f'_j d\mathbf{v}'. \quad (78)$$

Using the identity

$$\frac{\partial K^{\mu\nu}}{\partial v^\nu} = -2A_3 \frac{u^\mu}{u^3} = 2A_3 \frac{\partial}{\partial v^\mu} \left( \frac{1}{u} \right), \quad (79)$$

we can rewrite the friction term in the form

$$\eta^\mu = -2A_3 m \sum_j \frac{\partial h_j}{\partial v^\mu}(\mathbf{v}), \quad (80)$$

where

$$h(\mathbf{v}) = \int \frac{f(\mathbf{v}')}{|\mathbf{v} - \mathbf{v}'|} d\mathbf{v}'. \quad (81)$$

The functions  $g(\mathbf{v})$  and  $h(\mathbf{v})$  are called the Rosenbluth potentials. Using equations (76) and (80), we can readily check that the first and second moments of the velocity increments given by equations (14) and (17) return the expressions obtained by Rosenbluth et al. [14] and that the kinetic equation derived in [14] is equivalent to equation (74), although written in a different form. Therefore, we have recovered the kinetic equation of stellar dynamics directly from the Landau equation. This shows that the Landau equation is equivalent to the kinetic equation derived by Chandrasekhar [3] and Rosenbluth et al. [14] starting directly from the Fokker-Planck equation and evaluating the first and second moments of the velocity increments by considering a succession of binary encounters. This was expected because the Landau equation carries the same type of assumptions. However, it is surprising that the relation to the Landau equation was not mentioned in [3, 14, 15]. In particular, these authors write the Fokker-Planck equation in the form (13) with the diffusion coefficient in the second derivative ( $\partial_\mu \partial_\nu D^{\mu\nu}$ ) while a more symmetric form is the Landau equation (74) where the diffusion coefficient is inserted between the first derivatives ( $\partial_\mu D^{\mu\nu} \partial_\nu$ ), see equation (12). It is interesting to note, for historical reasons, that this symmetric form (from which we immediately deduce all the conservation laws of the system and the  $H$ -theorem [4]) has escaped to the study of stellar dynamicists [3, 14, 15] while the Landau equation was known long before in plasma physics.

#### 3.2 Diffusion and friction

We shall see that the Rosenbluth potentials can be easily calculated in  $d = 3$  when the field particles have an isotropic velocity distribution. The results of the previous section remain valid in  $d = 2$  provided that we replace  $A_3$  by  $A_2$  defined in equation (122). However, the calculation of the Rosenbluth potentials is apparently more complicated in  $d = 2$  than in  $d = 3$  and this is why we shall use another method in  $d = 2$  to obtain the diffusion and friction coefficients (see Sect. 4.1).

If the field particles have an isotropic velocity distribution, the Rosenbluth potentials in  $d = 3$  take the particularly simple form [14, 15, 39]:

$$h(v) = 4\pi \left[ \frac{1}{v} \int_0^v f(v_1) v_1^2 dv_1 + \int_v^{+\infty} f(v_1) v_1 dv_1 \right], \quad (82)$$

$$g(v) = \frac{4\pi v}{3} \left[ \int_0^v \left( 3v_1^2 + \frac{v_1^4}{v^2} \right) f(v_1) dv_1 + \int_v^{+\infty} \left( \frac{3v_1^3}{v} + vv_1 \right) f(v_1) dv_1 \right]. \quad (83)$$

Furthermore, when  $g = g(v)$  the diffusion tensor (76) can be put in the form of equation (18) with

$$D_{\parallel} = A_3 \sum_j m_j \frac{d^2 g_j}{dv^2}, \quad (84)$$

$$D_{\perp} = 2A_3 \sum_j m_j \frac{1}{v} \frac{dg_j}{dv}. \quad (85)$$

Using equation (83) we obtain

$$D_{\parallel} = \frac{8\pi}{3} A_3 \sum_j m_j \frac{1}{v} \left[ \int_0^v \frac{v_1^4}{v^2} f_j(v_1) dv_1 + v \int_v^{+\infty} v_1 f_j(v_1) dv_1 \right], \quad (86)$$

$$D_{\perp} = \frac{8\pi}{3} A_3 \sum_j m_j \frac{1}{v} \left[ \int_0^v \left( 3v_1^2 - \frac{v_1^4}{v^2} \right) f_j(v_1) dv_1 + 2v \int_v^{+\infty} v_1 f_j(v_1) dv_1 \right]. \quad (87)$$

On the other hand, when  $h = h(v)$  the friction term (80) can be written

$$\boldsymbol{\eta} = -2A_3 m \sum_j \frac{1}{v} \frac{dh_j}{dv} \mathbf{v}. \quad (88)$$

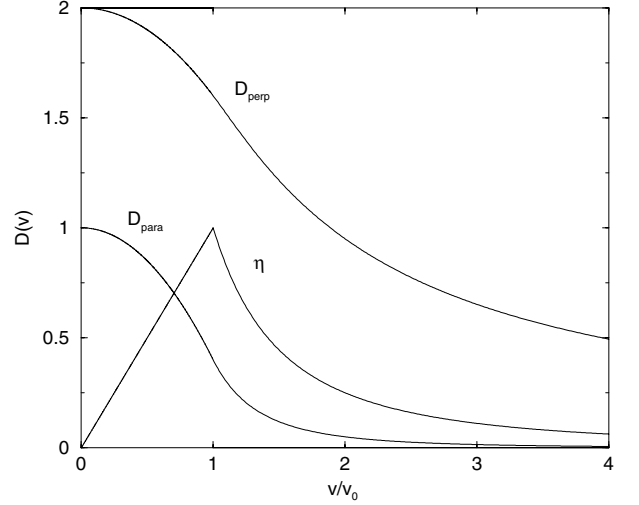
Using equation (82) we obtain

$$\boldsymbol{\eta} = 8\pi A_3 m \frac{\mathbf{v}}{v^3} \sum_j \int_0^v f_j(v_1) v_1^2 dv_1. \quad (89)$$

We note that these expressions are valid for *any* isotropic distribution of the field particles. If we substitute equations (86) and (89) into equation (21), we get the Fokker-Planck equation describing the evolution of a test particle in a bath with prescribed distribution  $f_j(v)$ . Alternatively, if we come back to the original Landau kinetic equation (6), assume an isotropic velocity distribution and substitute the general expressions (86) and (89) with now  $f_j = f_j(v, t)$  we obtain the integro-differential equation

$$\frac{\partial f_i}{\partial t} = 8\pi A_3 \sum_j \frac{1}{v^2} \frac{\partial}{\partial v} \left[ \frac{m_j}{3} \frac{\partial f_i}{\partial v} \left( \frac{1}{v} \int_0^v v_1^4 f_j(v_1, t) dv_1 + v^2 \int_v^{+\infty} v_1 f_j(v_1, t) dv_1 \right) + m_i f_i \int_0^v f_j(v_1, t) v_1^2 dv_1 \right], \quad (90)$$

describing the evolution of the system as a whole.



**Fig. 4.** Diffusion coefficients  $D_{\parallel}(v)/D_*$ ,  $D_{\perp}(v)/D_*$  and friction force  $\eta(v)/\eta_*$  as a function of  $v/v_0$  for a water-bag distribution of the bath in  $d = 3$ . The normalization factors are  $D_* = \frac{4\pi}{3} A_3 m_f \eta_0 v_0^2$  and  $\eta_* = \frac{8\pi}{3} A_3 m \eta_0 v_0$ .

### 3.3 Water-bag distribution

As an illustration of the previous formalism, we shall compute the diffusion coefficient and friction force of a test particle with mass  $m$  when the distribution function of the bath (composed of particles with mass  $m_f$ ) is a step function:  $f(\mathbf{v}) = \eta_0$  for  $v \leq v_0$  and  $f(\mathbf{v}) = 0$  for  $v > v_0$  with  $\eta_0 = 3\rho/(4\pi v_0^3)$  ( $\rho$  is the mass density of the field particles). Using equations (86), (87) and (89) we get for  $v \leq v_0$ :

$$D_{\parallel} = \frac{4\pi}{3} A_3 m_f \eta_0 \left( v_0^2 - \frac{3}{5} v^2 \right), \quad (91)$$

$$D_{\perp} = \frac{8\pi}{3} A_3 m_f \eta_0 \left( v_0^2 - \frac{v^2}{5} \right), \quad (92)$$

$$\eta = \frac{8\pi}{3} A_3 m \eta_0 v, \quad (93)$$

and for  $v > v_0$ :

$$D_{\parallel} = \frac{8\pi}{15} A_3 m_f \eta_0 v_0^5 \frac{1}{v^3}, \quad (94)$$

$$D_{\perp} = \frac{8\pi}{3} A_3 m_f \eta_0 \frac{1}{v} \left( v_0^3 - \frac{v_0^5}{5v^2} \right), \quad (95)$$

$$\eta = \frac{8\pi}{3} A_3 m \eta_0 \frac{v_0^3}{v^2}. \quad (96)$$

These quantities are plotted in Figure 4. We note that the friction term and the diffusion coefficient in the direction parallel to the direction of the test particle are related to each other by

$$\eta = \frac{2mD_{\parallel}}{m_f \left( v_0^2 - \frac{3}{5} v^2 \right)} v, \quad (v \leq v_0), \quad (97)$$

$$\eta = \frac{5mD_{\parallel}}{m_f v_0^2} v, \quad (v > v_0). \quad (98)$$

These expressions can be compared to the Einstein relation (32). We note that the role of the temperature is played here by  $m_f v_0^2$ .

Substituting these relations in equation (24) we get a Fokker-Planck equation (23) with an effective potential

$$U(v) = \frac{5}{3} \frac{m}{m_f} \left[ \frac{3}{2} - \ln \left( \frac{5}{2} - \frac{3}{2} \frac{v^2}{v_0^2} \right) \right], \quad (v \leq v_0), \quad (99)$$

$$U(v) = \frac{5}{2} \frac{m}{m_f} \frac{v^2}{v_0^2}, \quad (v > v_0). \quad (100)$$

The stationary solution of the Fokker-Planck equation is

$$P^{eq}(v) = A e^{-\frac{5}{2} \frac{m}{m_f} \left( \frac{5}{2} - \frac{3}{2} \frac{v^2}{v_0^2} \right)^{\frac{5m}{3m_f}}} \quad (v \leq v_0) \quad (101)$$

$$P^{eq}(v) = A e^{-\frac{5}{2} \frac{m}{m_f} \left( \frac{v}{v_0} \right)^2} \quad (v > v_0) \quad (102)$$

where  $A$  is a normalization constant. We explicitly check on this example that the stationary velocity distribution of the test particle is different from that of the bath. As explained at the end of Section 2.6, this is just a quasi-stationary distribution valid on a timescale  $(m_f/m)Nt_D \ll Nt_D$  for  $m \gg m_f$  because on longer timescales ( $\sim Nt_D$ ) the bath distribution will change.

We can also compute the diffusion coefficient and the friction force from equations (86), (87) and (89) for a Maxwellian velocity distribution of the field particles. In that case, we directly check that the Einstein relation (32) holds and that the diffusion coefficients are given by equations (45) and (46). For comparison with the water-bag model, we give below the asymptotic expressions of the diffusion coefficient and friction force for a thermal bath. For  $v \rightarrow +\infty$ , we have

$$D_{\parallel} = 2A_3 \frac{\rho}{\beta} \frac{1}{v^3}, \quad (103)$$

$$D_{\perp} = 2A_3 \rho m_f \frac{1}{v}, \quad (104)$$

$$\eta = 2A_3 \rho m \frac{1}{v^2}. \quad (105)$$

For  $v \rightarrow 0$ , we get

$$D_{\parallel} = \frac{4}{3} A_3 \left( \frac{\beta}{2\pi} \right)^{1/2} \rho m_f^{3/2} \left( 1 - \frac{3}{10} \beta m_f v^2 + \dots \right), \quad (106)$$

$$D_{\perp} = \frac{8}{3} A_3 \left( \frac{\beta}{2\pi} \right)^{1/2} \rho m_f^{3/2} \left( 1 - \frac{1}{10} \beta m_f v^2 + \dots \right), \quad (107)$$

$$\eta = \frac{4}{3} A_3 \left( \frac{\beta}{2\pi} \right)^{1/2} \rho m_f^{3/2} \beta m v. \quad (108)$$

Interestingly, these are the same asymptotic behaviors (same scaling) as in the water-bag model. For recent studies concerning the motion of a ‘‘test particle’’ in a thermal bath in relation with space plasmas, and for some numerical solutions of the corresponding Fokker-Planck equation see, e.g., Shizgal [42] and references therein.

## 4 The case $d = 2$

### 4.1 General expressions

We shall provide here general expressions of the diffusion coefficient and friction force in  $d = 2$  for any isotropic velocity distribution of the field particles. We shall use a method different from that exposed in Section 3 for  $d = 3$ . The following method also works in  $d = 3$  as shown in Appendix A.

Using equation (9), the diffusion coefficient (10) can be written

$$D^{\mu\nu} = \pi (2\pi)^d \sum_j m_j \int d\mathbf{k} d\mathbf{v}' k^\mu k^\nu \hat{u}(k)^2 \delta(\mathbf{k} \cdot \mathbf{u}) f_j(\mathbf{v}'). \quad (109)$$

Inserting the identity (40) in equation (109), we get

$$D^{\mu\nu} = (2\pi)^{2d} \sum_j m_j \int_0^{+\infty} dt \int d\mathbf{k} k^\mu k^\nu \hat{u}(k)^2 e^{i\mathbf{k} \cdot \mathbf{v}t} \hat{f}_j(\mathbf{k}t), \quad (110)$$

where we have introduced the Fourier transform

$$\hat{f}(\mathbf{k}) = \int f(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} \frac{d\mathbf{x}}{(2\pi)^d}. \quad (111)$$

For an isotropic velocity distribution,  $\hat{f}(\mathbf{k}) = \Phi(k)$  depends only on the modulus of  $\mathbf{k}$  (see the explicit expression below). Setting  $\tau = kt$  we finally obtain

$$D^{\mu\nu} = (2\pi)^{2d} \sum_j m_j \int_0^{+\infty} d\tau \int d\mathbf{k} \frac{k^\mu k^\nu}{k} \hat{u}(k)^2 e^{i\hat{\mathbf{k}} \cdot \mathbf{v}\tau} \Phi_j(\tau) \quad (112)$$

where  $\hat{\mathbf{k}} = \mathbf{k}/k$ . This expression is valid in  $d$  dimensions. We now specialize on the case  $d = 2$ . Introducing polar coordinates with the  $x$ -axis in the direction of  $\mathbf{v}$  and using the identities

$$\int_0^{2\pi} e^{ix \cos \theta} d\theta = 2\pi J_0(x), \quad (113)$$

$$\int_0^{2\pi} e^{ix \cos \theta} \sin^2 \theta d\theta = 2\pi \frac{J_1(x)}{x}, \quad (114)$$

we find that  $D^{\mu\nu}$  is given by equation (18) with

$$D_{\parallel} = (2\pi)^5 \sum_j m_j \frac{1}{v} \int_0^{+\infty} k^2 \hat{u}(k)^2 dk \times \int_0^{+\infty} dx \left[ J_0(x) - \frac{J_1(x)}{x} \right] \Phi_j \left( \frac{x}{v} \right), \quad (115)$$

$$D_{\perp} = (2\pi)^5 \sum_j m_j \frac{1}{v} \int_0^{+\infty} k^2 \hat{u}(k)^2 dk \times \int_0^{+\infty} dx \frac{J_1(x)}{x} \Phi_j \left( \frac{x}{v} \right). \quad (116)$$

Now, in  $d = 2$ ,

$$\Phi(k) = \frac{1}{2\pi} \int_0^{+\infty} f(v_1) J_0(kv_1) v_1 dv_1. \quad (117)$$

Substituting this expression in equations (115) and (116) and introducing the notations

$$I_{\parallel}(\lambda) = \int_0^{+\infty} \left[ J_0(x) - \frac{J_1(x)}{x} \right] J_0(\lambda x) dx, \quad (118)$$

$$I_{\perp}(\lambda) = \int_0^{+\infty} \frac{J_1(x)}{x} J_0(\lambda x) dx, \quad (119)$$

we obtain

$$D_{\parallel} = 2\pi A_2 \sum_j m_j \frac{1}{v} \int_0^{+\infty} f_j(v_1) I_{\parallel} \left( \frac{v_1}{v} \right) v_1 dv_1, \quad (120)$$

$$D_{\perp} = 2\pi A_2 \sum_j m_j \frac{1}{v} \int_0^{+\infty} f_j(v_1) I_{\perp} \left( \frac{v_1}{v} \right) v_1 dv_1, \quad (121)$$

where

$$A_2 = 8\pi^3 \int_0^{+\infty} k^2 \hat{u}(k)^2 dk. \quad (122)$$

Finally, the friction is given by equation (20). Note that these expressions are valid for an arbitrary isotropic velocity distribution of the field particles. They remain true if the distribution function depends on time. Therefore, by substituting these results in the Landau equation (6) we obtain a self-consistent kinetic equation for an isotropic distribution function  $f(v, t)$  which is the counterpart in  $d = 2$  of equation (90) in  $d = 3$ . However, because it involves complicated functions (118) and (119), its expression is less explicit.

## 4.2 Water-bag distribution

The previous relations are quite general. Let us now consider for illustration the case where the distribution function of the field particles (with mass  $m_f$ ) is a step function:  $f(v) = \eta_0$  for  $v \leq v_0$  and  $f(v) = 0$  for  $v > v_0$  where  $\eta_0 = \rho/(\pi v_0^2)$ . Using the identity

$$xJ_0(x) = [xJ_1(x)]', \quad (123)$$

the Fourier transform (117) of the distribution function is

$$\Phi(\xi) = \frac{\eta_0 v_0}{2\pi \xi} J_1(\xi v_0). \quad (124)$$

Inserting this expression in equations (115) and (116), we obtain

$$D_{\parallel} = 2\pi A_2 m_f \eta_0 v_0 R_{\parallel} \left( \frac{v_0}{v} \right), \quad (125)$$

$$D_{\perp} = 2\pi A_2 m_f \eta_0 v_0 R_{\perp} \left( \frac{v_0}{v} \right), \quad (126)$$

where we have introduced the functions

$$R_{\parallel}(\lambda) = \int_0^{+\infty} dx \left[ J_0(x) - \frac{J_1(x)}{x} \right] \frac{1}{x} J_1(\lambda x), \quad (127)$$

$$R_{\perp}(\lambda) = \int_0^{+\infty} dx \frac{J_1(x)}{x^2} J_1(\lambda x). \quad (128)$$

Using equation (20), the friction term is given by

$$\eta = -\frac{m}{m_f} \left[ \frac{dD_{\parallel}}{dv} + \frac{1}{v} (D_{\parallel} - D_{\perp}) \right]. \quad (129)$$

The integrals (127) and (128) can be expressed in terms of hypergeometric functions. Their asymptotic behaviors are derived in Appendix B. Using these results, we obtain the asymptotic behaviors of the diffusion coefficients and friction force. For  $v \rightarrow +\infty$ , we get

$$D_{\parallel} = \frac{1}{4} \pi A_2 m_f \eta_0 v_0^4 \frac{1}{v^3}, \quad (130)$$

$$D_{\perp} = \pi A_2 m_f \eta_0 v_0^2 \frac{1}{v}, \quad (131)$$

$$\eta = \pi A_2 m \eta_0 v_0^2 \frac{1}{v^2}. \quad (132)$$

We note that the leading term  $\sim v^{-2}$  of the friction  $\eta$  is due to the diffusion in the direction perpendicular to the velocity of the test particle, i.e. the term  $D_{\perp}$  in equation (129). The contribution of the diffusion coefficient in the direction parallel to the velocity of the test particle decreases more rapidly like  $v^{-4}$ . For  $v \rightarrow 0$ , we get

$$D_{\parallel} = \pi A_2 m_f \eta_0 v_0 \left[ 1 - \frac{3}{8} \left( \frac{v}{v_0} \right)^2 + \dots \right], \quad (133)$$

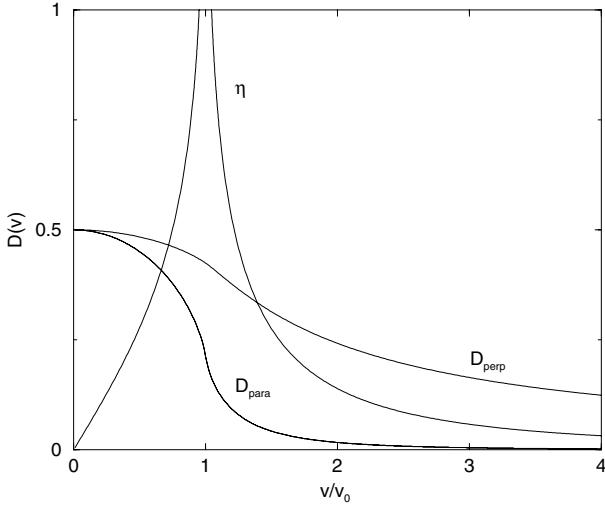
$$D_{\perp} = \pi A_2 m_f \eta_0 v_0 \left[ 1 - \frac{1}{8} \left( \frac{v}{v_0} \right)^2 + \dots \right], \quad (134)$$

$$\eta = \pi m A_2 \eta_0 \frac{v}{v_0}. \quad (135)$$

On the other hand, using equations (129) and (219), we find that the friction coefficient diverges for  $v \rightarrow v_0$  like

$$\eta = 2A_2 m \eta_0 (2 - 2\gamma - \ln 2 - 2\psi(3/2) - \ln |1 - v/v_0|). \quad (136)$$

The diffusion coefficients and friction force for a water-bag distribution of the bath are plotted in Figure 5.



**Fig. 5.** Diffusion coefficients  $D_{\parallel}(v)/D_*$ ,  $D_{\perp}(v)/D_*$  and friction force  $\eta(v)/\eta_*$  as a function of  $v/v_0$  for a water bag distribution of the bath in  $d = 2$ . The normalization factors are  $D_* = 2\pi A_2 m_f \eta_0 v_0$  and  $\eta_* = (m/m_f)(D_*/v_0)$ .

We can also use the previous formalism in the case where the distribution of the bath is Maxwellian (thermal bath), in which case

$$\Phi_i(\xi) = \frac{\rho_i}{(2\pi)^2} e^{-\frac{\xi^2}{2\beta m_i}}. \quad (137)$$

Substituting this expression in equations (115) and (116) and carrying out the integrations, we can show that they return the results (49) and (50) obtained by a different method. For comparison with the water-bag model, we give below the asymptotic expressions of the diffusion coefficient and friction force for a thermal bath. For  $v \rightarrow +\infty$ , we have

$$D_{\parallel} = A_2 \frac{\rho}{\beta} \frac{1}{v^3}, \quad (138)$$

$$D_{\perp} = A_2 \rho m_f \frac{1}{v}, \quad (139)$$

$$\eta = A_2 \rho m_f \frac{1}{v^2}. \quad (140)$$

For  $v \rightarrow 0$ , we get

$$D_{\parallel} = \frac{\pi}{2} A_2 \left(\frac{\beta}{2\pi}\right)^{1/2} \rho m_f^{3/2} \left(1 - \frac{3}{8} \beta m_f v^2 + \dots\right), \quad (141)$$

$$D_{\perp} = \frac{\pi}{2} A_2 \left(\frac{\beta}{2\pi}\right)^{1/2} \rho m_f^{3/2} \left(1 - \frac{1}{8} \beta m_f v^2 + \dots\right), \quad (142)$$

$$\eta = \frac{\pi}{2} A_2 \left(\frac{\beta}{2\pi}\right)^{1/2} \rho m_f^{3/2} \beta m v. \quad (143)$$

Interestingly, these are the same asymptotic behaviors (same scaling) as in the water-bag model. This is also the case in  $d = 3$  (see Sect. 3).

### 4.3 Example: Coulombian plasma

In a recent paper, Benedetti et al. [23] have considered a 2D model of Coulomb oscillators interacting via a potential of the form

$$u_{ij} = -\frac{\xi}{N} \ln |\mathbf{r}_i - \mathbf{r}_j|. \quad (144)$$

They have calculated the friction force  $\mathbf{K} = \langle (\Delta \mathbf{v}) \rangle / \Delta t$  experienced by a test particle by evaluating the average variation of the velocity caused by a succession of binary encounters. The calculations are relatively difficult because the explicit expression of the differential cross section for a Coulombian interaction is not known in  $d = 2$ . We shall reconsider this problem with our approach based on the Landau or Lenard-Balescu equation which does not require the explicit expression of the differential cross section, but only the Fourier transform of the potential of interaction. As noted after equation (74), for weakly interacting systems, the form of potential just determines the timescale of relaxation through the constant  $A$ .

First, we note that in  $d = 2$  the potential (144) is solution of the differential equation

$$\Delta u = -\frac{2\pi\xi}{N} \delta(\mathbf{r}). \quad (145)$$

This immediately leads to

$$(2\pi)^2 \hat{u}(k) = \frac{2\pi\xi}{N k^2}. \quad (146)$$

Using equation (146), we find that the constant (122) is given by

$$A_2 = \frac{2\pi\xi^2}{N^2} A \quad (147)$$

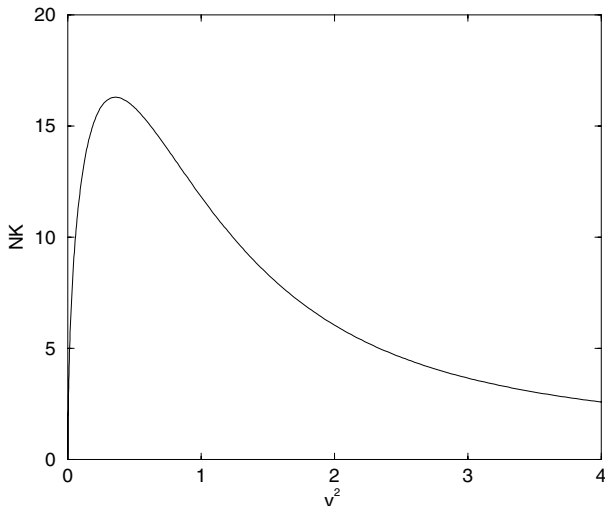
where

$$A = \int_0^{+\infty} \frac{dk}{k^2}. \quad (148)$$

We note that for a Coulombian potential in  $d = 2$ , the Coulomb factor  $A$  diverges *linearly* with the distance for  $\lambda = 2\pi/k \rightarrow +\infty$ . This contrasts from the 3D case where the divergence is only logarithmic. In paper [22], we have suggested that this divergence would be cured (as in the 3D case) by using the complete form of the Lenard-Balescu equation including the collective effects encapsulated in the dielectric function. These calculations are completed in Appendix C. However, in a first approach, we shall neglect collective effects and argue phenomenologically that  $A$  should scale like the Debye length in 2D. Note that there can be a numerical factor between  $A$  and the Debye length so that our approach will only provide an estimate of the diffusion coefficient.

If we specialize on the case where the distribution of the bath is a step function, we can use the results of Section 4.2 with

$$\eta_0 = \frac{N}{(2\pi)^2 k_B T R^2}, \quad v_0 = 2\sqrt{k_B T}, \quad (149)$$



**Fig. 6.** Friction force as a function of the velocity for an isothermal distribution of the bath (theoretical curve obtained with our approach). For comparison, we have adopted the same normalization as in [23].

where, following [23], we have introduced a typical radius via  $\rho = N/(\pi R^2)$  and defined the “temperature” via  $\langle v^2 \rangle = 2k_B T$  (we have taken  $m = 1$ ). Recalling that for identical particles the friction force is given by  $K = 2\eta$  and using the asymptotic results (132) and (135), we get for  $v \rightarrow +\infty$ :

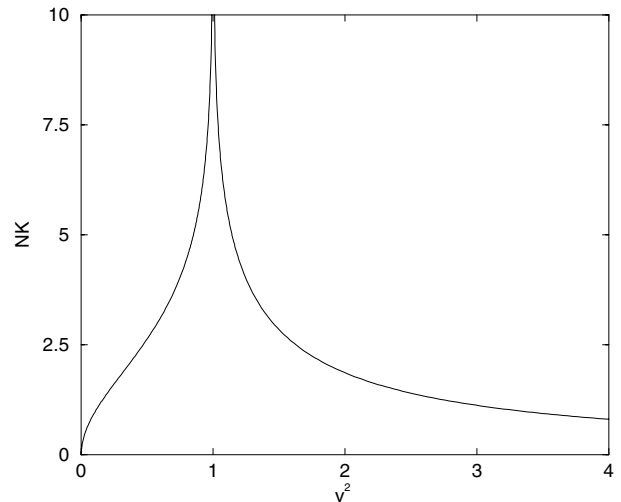
$$K = 4 \frac{\xi^2}{NR^2} \Lambda \frac{1}{v^2}, \quad (150)$$

and for  $v \rightarrow 0$ :

$$K = \frac{1}{2} \frac{\xi^2}{NR^2} \frac{\Lambda}{(k_B T)^{3/2}} v. \quad (151)$$

These asymptotic results agree with those obtained by Benedetti et al. [23] with a different method, up to a factor  $2(\pi - 2)/\pi \simeq 0.727\dots$ . Owing to the remark following equation (148), we should not give too much credit on the exact value of the numerical constant (see Appendix C for a more precise determination of the value of  $\Lambda$ ). However, it would be important in future works to determine whether our approach and that of [23] are really equivalent or not.

We emphasize that our approach provides the expression of the diffusion tensor  $D^{\mu\nu}$  and friction force  $\eta^\mu$  for any value of the velocity; these quantities have not been calculated in [23]. When the distribution of the field particles is Maxwellian (thermal bath), they are given by the analytical formulae (49) and (50) obtained in [22]. The friction force obtained from equation (32) is plotted as a function of the velocity in Figure 6. We have multiplied our expression by the factor  $2(\pi - 2)/\pi$ . In that case, our analytical formula of  $K$  matches remarkably well, in the whole range of velocities, the curve obtained numerically by [23]. For a water-bag distribution of the bath, the diffusion coefficients and friction force can be expressed



**Fig. 7.** Friction force as a function of the velocity when the distribution of the bath is a step function (theoretical curve obtained with our approach). For comparison, we have adopted the same normalization as in [23]. For  $N \rightarrow +\infty$ , the friction force diverges logarithmically at the velocity  $v_0$ .

in terms of integrals of Bessel functions (125)–(129) or, equivalently, in terms of Hypergeometric functions using equations (211), (212) and (218). The friction force obtained from equation (129) is plotted as a function of the velocity in Figure 7. Again, we have multiplied our expression by the factor  $2(\pi - 2)/\pi$  and we get an excellent agreement with the numerical results of [23]. We note that in our approach valid for  $N \rightarrow +\infty$ , the friction force *diverges* for  $v \rightarrow v_0$ . Using equation (136), we find that the divergence is like

$$K \sim 2 \frac{\xi^2}{NR^2} \frac{\Lambda}{k_B T} \left( a - b \ln \left| 1 - \frac{v}{2\sqrt{k_B T}} \right| \right), \quad (152)$$

where  $a = (2 - 2\gamma - \ln 2 - 2\psi(3/2))/\pi \simeq 0.025287$  and  $b = 1/\pi \simeq 0.3183$ . This behavior is consistent with the results of [23] who obtain the  $K - v$  curve from  $N$ -body simulations and find that the maximum velocity increases as  $N$  increases. For finite  $N$  systems, there is no singularity. The singularity appears for  $N \rightarrow +\infty$ . Formally, the curve of Figure 7 and the divergence of the friction force at  $v = v_0$  when  $N \rightarrow +\infty$  share some analogies with the divergence of the specific heats in second order phase transitions (e.g., the curve of Fig. 7 looks similar to the  $\lambda$ -transition in superfluid helium).

#### 4.4 Statistics of fluctuations

In order to better understand the previous results, and in particular the linear divergence of the diffusion coefficient, we can analyze the statistics of fluctuations of the force produced by a random distribution of charges in  $d = 2$ . The force created by the charges at some point of origin

is given by

$$\mathbf{F} = \sum_{i=1}^N \frac{\xi}{N} \frac{\mathbf{r}_i}{r_i^2}. \quad (153)$$

We shall assume that the charges are randomly distributed with uniform distribution. In that case, the force  $\mathbf{F}$  fluctuates and we have to determine its distribution  $W(\mathbf{F})$ . Formally, this mathematical problem is the same as the one considered by Chavanis and Sire [35] in their investigation of the statistics of the fluctuations of velocity  $\mathbf{V}$  created by a random distribution of point vortices in  $d = 2$ . Therefore, we can apply their results just by making the substitution  $\mathbf{F} \leftrightarrow \mathbf{V}$  (we also note that to leading order  $\langle F^2 \rangle \sim N \times (1/N^2) \sim 1/N$  so that the proper scaled variable is  $x = FN^{1/2}$ ). To have exactly the same notations, we set  $\xi/N = \gamma/(2\pi)$ . It is shown in [35] that the distribution of  $\mathbf{F}$  is a marginal Gaussian distribution

$$W(\mathbf{F}) = \frac{4}{n\gamma^2 \ln N} e^{-\frac{4\pi}{n\gamma^2 \ln N} F^2} \quad (F \ll F_{crit}(N)), \quad (154)$$

$$W(\mathbf{F}) = \frac{n\gamma^2}{4\pi^2 F^4} \quad (F \gg F_{crit}(N)), \quad (155)$$

$$F_{crit}(N) \sim \left( \frac{n\gamma^2}{4\pi} \ln N \right)^{1/2} \ln^{1/2}(\ln N), \quad (156)$$

where  $n$  is the spatial distribution of the particles assumed to be homogeneous. This distribution is intermediate between Normal and Lévy laws: the core of the distribution is Gaussian but the tail decreases algebraically as  $F^{-4}$ . It is dominated by the contribution of the nearest neighbor [35]. Note that the variance of the scaled variable  $x = FN^{1/2}$  diverges logarithmically with  $N$  since

$$\langle F^2 \rangle = \frac{n\gamma^2}{4\pi} \ln N. \quad (157)$$

Analytical results for the spatial correlations of the force are derived in [36]. Here, we shall concentrate on the basics in order to make the connection with the kinetic theory developed previously. If we neglect collective effects, the spatial correlation function of the force can be written (see Eq. (100) of Ref. [21]):

$$\langle \mathbf{F}(\mathbf{0}) \cdot \mathbf{F}(\mathbf{r}) \rangle = (2\pi)^2 n \int k^2 \hat{u}(k)^2 e^{-i\mathbf{k} \cdot \mathbf{r}} d\mathbf{k}. \quad (158)$$

Introducing a polar system of coordinates with the  $x$  axis in the direction of  $\mathbf{r}$ , using  $(2\pi)^2 \hat{u}(k) = \gamma/k^2$  and introducing a large scale cut-off  $\Lambda$ , we obtain

$$\langle \mathbf{F}(\mathbf{0}) \cdot \mathbf{F}(\mathbf{r}) \rangle = \frac{n\gamma^2}{2\pi} \int_{1/\Lambda}^{+\infty} \frac{J_0(kr)}{k} dk, \quad (159)$$

which behaves like

$$\langle \mathbf{F}(\mathbf{0}) \cdot \mathbf{F}(\mathbf{r}) \rangle \simeq \frac{n\gamma^2}{2\pi} \ln \left( \frac{\Lambda}{r} \right), \quad (160)$$

for  $r/\Lambda \rightarrow 0$ . An alternative derivation of this result is given in reference [36], Appendix D. In particular, we find that the correlation function diverges logarithmically for  $r \rightarrow 0$ , in agreement with the result (157) [note that in  $d = 2$  the length scales as  $L \sim (N/n)^{1/2}$  which yields the factor  $\frac{1}{2} \ln N$ ]. We shall now see how collective effects can remove this logarithmic divergence. The expression (158) neglects the correlations between particles. Now, applying equation (51) of reference [21] to the present context, it is found that the spatial correlation function is solution of the differential equation

$$\Delta h - k_D^2 h = \beta\gamma \delta(\mathbf{x}), \quad (161)$$

where  $k_D = (\beta n\gamma)^{1/2}$  is the Debye wavenumber. The spatial correlation is then given by

$$h(x) = -\frac{\beta\gamma}{2\pi} K_0(k_D x), \quad (162)$$

which is the Debye-Hückel result in 2D. The Fourier transform of the correlation function can be written

$$(2\pi)^2 n \hat{h}(k) = -\frac{k_D^2}{k^2 + k_D^2}, \quad (163)$$

and the correlational energy (see Eq. (57) of paper [21]) can be written  $W_c = -n\gamma V/(4\pi)[\gamma_E + (1/2) \ln(\beta n\gamma) - \ln 2]$  where  $\gamma_E = 0.577\dots$  is the Euler constant. If we account for collective effects in the computation of the force autocorrelation function, we obtain (see Eq. (96) of Ref. [21]):

$$\langle \mathbf{F}(\mathbf{0}) \cdot \mathbf{F}(\mathbf{r}) \rangle = (2\pi)^2 n \int \frac{k^2 \hat{u}(k)^2}{1 + (2\pi)^2 \beta n \hat{u}(k)} e^{-i\mathbf{k} \cdot \mathbf{r}} d\mathbf{k}, \quad (164)$$

where the new terms arise because of the correlations encapsulated in the function  $\hat{h}(k)$ . Thus, combining the previous results, we get

$$\langle \mathbf{F}(\mathbf{0}) \cdot \mathbf{F}(\mathbf{r}) \rangle = \frac{n\gamma^2}{2\pi} \int_0^{+\infty} \frac{J_0(kr)}{k} \frac{k^2}{k^2 + k_D^2} dk, \quad (165)$$

where there is no need to introduce an *ad hoc* large-scale cut-off anymore. We now obtain the result

$$\langle \mathbf{F}(\mathbf{0}) \cdot \mathbf{F}(\mathbf{r}) \rangle = \frac{n\gamma^2}{2\pi} K_0(k_D r). \quad (166)$$

For  $k_D r \rightarrow 0$ , we recover equation (160) with  $\Lambda = k_D^{-1}$ . Therefore, the divergence in equation (159) can be regularized by properly accounting for correlations among the particles. For comparison, the spatial correlation of the force in 3D within the mean-field Debye-Hückel theory is

$$\begin{aligned} \langle \mathbf{F}(\mathbf{0}) \cdot \mathbf{F}(\mathbf{r}) \rangle &= n\gamma^2 \int_0^{+\infty} \frac{\sin(kr)}{kr} \frac{k^2}{k^2 + k_D^2} dk \\ &= \frac{\pi}{2} n\gamma^2 \frac{1}{r} e^{-k_D r}, \end{aligned} \quad (167)$$

where  $k_D = (4\pi\beta n e^2)^{1/2}$  is the Debye shielding length in  $d = 3$ . Note that the integral remains well-behaved if we



neglect collective effects ( $k_D = 0$ ) contrary to the case  $d = 2$ . On the other hand, the fact that the correlation function diverges as  $1/r$  for  $r \rightarrow 0$  is due to the fact that the distribution  $W(\mathbf{F})$  is a particular Lévy law, called the Holtzmark distribution, whose variance is infinite [9].

Let us now consider the temporal correlations of the force. If we neglect collective effects, the correlation function is given by (see Eq. (93) of Ref. [22]):

$$\langle \mathbf{F}(0) \cdot \mathbf{F}(t) \rangle = (2\pi)^2 n \int k^2 \hat{u}(k)^2 e^{-i\mathbf{k} \cdot \mathbf{v}t} e^{-k^2 t^2 / (2\beta)} d\mathbf{k}. \quad (168)$$

Using the same procedure as before, we get

$$\langle \mathbf{F}(0) \cdot \mathbf{F}(t) \rangle = \frac{n\gamma^2}{2\pi} \int_{vt/\Lambda}^{+\infty} \frac{J_0(x)}{x} e^{-x^2/(2\beta v^2)} dx, \quad (169)$$

which behaves like

$$\langle \mathbf{F}(0) \cdot \mathbf{F}(t) \rangle = \frac{n\gamma^2}{2\pi} \ln \left( \frac{\Lambda}{vt} \right), \quad (170)$$

for  $vt/\Lambda \rightarrow 0$ . The previous expression indicates that the correlation of the force is almost independent of time. This implies that the diffusion coefficient, calculated with the Kubo formula

$$D = \int_0^{+\infty} \langle \mathbf{F}(0) \cdot \mathbf{F}(t) \rangle dt \sim \int_0^{+\infty} dt, \quad (171)$$

diverges *linearly* with time. This temporal divergence is the analogue of the spatial divergence  $\Lambda \sim \int_0^{+\infty} d\lambda$  obtained in the kinetic approach of Section 4.3. The same effect occurs for gravitational systems in  $d = 3$  where the *logarithmic* divergence of the diffusion coefficient can be viewed either as a spatial  $\sim \int dk/k$  or temporal  $\int dt/t$  divergence (see Sect. 2.9 of [22]). Now, if we take into account collective effects, the temporal correlation function is given by (see Eq. (98) of Ref. [22]):

$$\langle \mathbf{F}(0) \cdot \mathbf{F}(t) \rangle = (2\pi)^2 \int \frac{k^2 \hat{u}(k)^2}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}_1)|^2} e^{i\mathbf{k} \cdot \mathbf{u}t} f(\mathbf{v}_1) d\mathbf{v}_1 d\mathbf{k}. \quad (172)$$

By investigating the poles of the integrand (for a Maxwellian distribution), it is found that the contribution of each mode  $k$  should decay with time exponentially rapidly (modulated by an oscillatory factor) with an exponent  $\gamma_k$  corresponding to the Landau damping rate. This is the imaginary part of the pulsation  $\omega$  which cancels out the dielectric function  $\epsilon(\mathbf{k}, \omega)$ . Therefore, collective effects modify the expression of the temporal correlation function and, consequently, of the diffusion coefficient. Indeed, according to the Kubo formula, the diffusion coefficient entering in the Lenard-Balescu equation

$$D^{\mu\nu} = \pi(2\pi)^2 \int d\mathbf{v}_1 d\mathbf{k} k^\mu k^\nu \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}_1)|^2} \delta(\mathbf{k} \cdot \mathbf{u}) f(\mathbf{v}_1), \quad (173)$$

is the time integral of the correlation function  $\langle F^\mu(0) F^\nu(t) \rangle$ . When the dielectric function is taken into

account, it is seen that the integrals on  $k$  in equation (173) are now convergent (see discussion in Sec. 2.8.1 of reference [22]). By these means, we can provide a justification of the upper cut-off which appears in equation (148). The calculations are detailed in Appendix C.

## 5 The case $d = 1$

For  $d = 1$ , the Fokker-Planck equation (7) reduces to

$$\frac{\partial P}{\partial t} = (2\pi)^2 \frac{\partial}{\partial v} \int_{-\infty}^{+\infty} dv' \int_0^{+\infty} dk k^2 \frac{\hat{u}(k)^2}{|\epsilon(k, kv)|^2} \delta[k(v-v')] \times \sum_j \left( m_j f'_j \frac{\partial P}{\partial v} - m P \frac{df'_j}{dv} \right). \quad (174)$$

Using  $\delta[k(v-v')] = (1/|k|)\delta(v-v')$ , the integral over  $v'$  is straightforward and yields

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v} A(v) \sum_j \left( m_j f_j \frac{\partial P}{\partial v} - m P \frac{df_j}{dv} \right), \quad (175)$$

where

$$A(v) = (2\pi)^2 \int_0^{+\infty} dk \frac{k \hat{u}(k)^2}{|\epsilon(k, kv)|^2}. \quad (176)$$

This can be rewritten in the form

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v} \left( D \frac{\partial P}{\partial v} + P \eta \right), \quad (177)$$

with

$$D(v) = A(v) \sum_j m_j f_j(v), \quad (178)$$

and

$$\eta(v) = -mA(v) \sum_j \frac{df_j}{dv}. \quad (179)$$

If the field particles all have the same mass  $m_f$ , these expressions simplify in

$$D(v) = A(v) m_f f(v), \quad \eta(v) = -A(v) m_f f'(v). \quad (180)$$

The effective potential (24) is

$$U(v) = -\frac{m}{m_f} \ln f(v), \quad (181)$$

and the Fokker-Planck equation can be rewritten

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v} \left[ D(v) \left( \frac{\partial P}{\partial v} - P \frac{m}{m_f} \frac{d \ln f}{dv} \right) \right], \quad (182)$$

with

$$D(v) = (2\pi)^2 m_f f(v) \int_0^{+\infty} \frac{k \hat{u}(k)^2}{|\epsilon(k, kv)|^2} dk. \quad (183)$$

We note that the stationary solution of the Fokker-Planck equation (182) is

$$P^{eq}(v) = C f^{\frac{m}{m_f}}(v), \quad (184)$$

and that  $P^{eq}(v) = f(v)$  when the mass of the test particle is equal to the mass of the field particles  $m = m_f$ . More explicit expressions of the diffusion coefficient valid for a thermal bath are given in [22].

## 6 Conclusion

In this paper, we have discussed the kinetic theory of systems with long-range interactions in a relatively unified framework starting from the Landau and Lenard-Balescu equations. Using a test particle approach, we have given explicit expressions for the diffusion coefficient and friction force entering in the Fokker-Planck equation for different potentials of interaction in different dimensions of space and for different distributions of the bath. We have also considered the possibility of a distribution of mass among the particles and shown how the results (in particular the Einstein relation) are modified in that case.

For Coulombian and Newtonian potentials in  $d = 3$ , we have enlightened the connection between results of plasma physics and results of stellar dynamics which have been derived almost independently and in a relatively different form. We have shown that the kinetic equation derived in stellar dynamics by Chandrasekhar [3] and Rosenbluth et al. [14] from the Fokker-Planck equation can also be obtained from the Landau [2] equation of plasma physics. We have then considered the extension of these kinetic theories in  $d = 2$ . For a Maxwellian distribution of the bath, the diffusion and friction coefficients can be calculated explicitly in terms of Bessel functions as shown in [22]. In the present paper, we have generalized our approach to an arbitrary isotropic distribution of the bath and expressed the results in terms of integrals of Bessel functions. In the case where the distribution function of the field particles is a step function, we have shown that the asymptotic expressions of our obtained diffusion and friction coefficients reproduce those found by Benedetti et al. [23] (up to a factor  $2(\pi - 2)/\pi$ ) with a different method. More generally, our analytical expressions give a good agreement with their numerical results for all velocities. We have shown analytically that, for  $N \rightarrow +\infty$ , the friction diverges logarithmically at the critical velocity  $v_0$  (while for finite  $N$  there is no divergence). We have also shown that the linear divergence of the diffusion coefficient resulting from the Landau approximation could be removed by considering collective effects as in the Lenard-Balescu treatment of a 3D plasma. Finally, we have shown how the results of kinetic theory simplify in  $d = 1$ .

For future perspectives, it could be mentioned that the results presented in this paper can be formally extended to a generalized class of kinetic equations, associated with a generalized thermodynamical framework, introduced by Kaniadakis [43] and Chavanis [44,39]. These generalized equations could be justified in the case of complex systems

when the transition probabilities from one state to the other are different from the form that is usually considered due to microscopic constraints (“hidden constraints”) that affect the dynamics. However, the domains of application of these generalized kinetic theories remains to be better specified.

## Appendix A: Alternative derivation of the diffusion coefficients and friction force in $d = 3$

In Section 3, we have derived the expressions of the diffusion coefficients and friction force for an isotropic velocity distribution of the field particles in  $d = 3$  by using the Rosenbluth potentials. In this Appendix, we show that we can obtain the same results by using the method developed in Section 4 which extends to  $d = 2$ .

Starting from the general expression (112) of the diffusion coefficient and introducing a spherical system of coordinates with the  $z$ -axis in the direction of  $\mathbf{v}$ , we find after straightforward algebra, that the diffusion tensor can be written as in equation (18) with

$$D_{\parallel} = (2\pi)^7 \sum_j m_j \frac{1}{v} \int_0^{+\infty} k^3 \hat{u}(k)^2 dk \times \int_0^{+\infty} dx \left[ 2 \left( 1 - \frac{2}{x^2} \right) \frac{\sin x}{x} + 4 \frac{\cos x}{x^2} \right] \Phi_j \left( \frac{x}{v} \right), \quad (185)$$

$$D_{\perp} = (2\pi)^7 \sum_j m_j \frac{1}{v} \int_0^{+\infty} k^3 \hat{u}(k)^2 dk \times \int_0^{+\infty} dx \left[ 4 \frac{\sin x}{x^3} - 4 \frac{\cos x}{x^2} \right] \Phi_j \left( \frac{x}{v} \right). \quad (186)$$

The Fourier transform of an isotropic velocity distribution of the field particles in  $d = 3$  is

$$\Phi(k) = \frac{1}{2\pi^2 k} \int_0^{+\infty} f(v_1) \sin(kv_1) v_1 dv_1. \quad (187)$$

Substituting this expression in equations (185) and (186) and introducing the functions

$$I_{\parallel}(\lambda) = \int_0^{+\infty} dx \left[ 2 \left( 1 - \frac{2}{x^2} \right) \frac{\sin x}{x} + 4 \frac{\cos x}{x^2} \right] \frac{1}{x} \sin(\lambda x), \quad (188)$$

$$I_{\perp}(\lambda) = \int_0^{+\infty} dx \left[ 4 \frac{\sin x}{x^3} - 4 \frac{\cos x}{x^2} \right] \frac{1}{x} \sin(\lambda x), \quad (189)$$

we obtain

$$D_{\parallel} = 8A \sum_j m_j \int_0^{+\infty} f_j(v_1) I_{\parallel} \left( \frac{v_1}{v} \right) v_1 dv_1. \quad (190)$$

$$D_{\perp} = 8A \sum_j m_j \int_0^{+\infty} f_j(v_1) I_{\perp} \left( \frac{v_1}{v} \right) v_1 dv_1. \quad (191)$$

The integrals (188) and (189) can be calculated explicitly. Using the results:  $I_{\parallel}(\lambda) = \pi/3$ ,  $I_{\perp}(\lambda) = 2\pi/3$  for  $\lambda > 1$  and  $I_{\parallel}(\lambda) = (\pi/3)\lambda^3$ ,  $I_{\perp}(\lambda) = \pi\lambda - (\pi/3)\lambda^3$  for  $\lambda < 1$ , we recover the expressions (86) and (87). The friction  $\eta$  can be obtained from equation (20) and we recover equation (89).

## Appendix B: Asymptotic behaviors of the functions (127) and (128)

### B.1 Asymptotic behaviors for $\lambda \rightarrow 0$

In this Appendix, we determine the asymptotic behaviors of the functions (127) and (128) for  $\lambda \rightarrow 0$ . Using identities (113) and (114), equation (127) can be rewritten

$$R_{\parallel}(\lambda) = \frac{1}{\pi} \int_0^{\pi} d\theta \cos^2 \theta \int_0^{+\infty} e^{ix \cos \theta} \frac{J_1(\lambda x)}{x} dx. \quad (192)$$

Setting  $t = \cos \theta$ , we obtain

$$R_{\parallel}(\lambda) = \frac{1}{\pi} \int_{-1}^{+1} \frac{t^2 dt}{\sqrt{1-t^2}} \int_0^{+\infty} e^{ixt} \frac{J_1(\lambda x)}{x} dx. \quad (193)$$

In this expression,  $t$  and  $x$  are real and the domains of integration  $\tau_0$ :  $-1 \leq t \leq 1$  and  $\zeta_0$ :  $0 \leq x < +\infty$  are taken along the real axis. Under these circumstances, we cannot simply expand the last function in Taylor series for  $\lambda \rightarrow 0$  because the resulting integrals would not be convergent. However, regarding  $x$  and  $t$  as complex variables, it is possible to choose paths of integration along which this expansion will converge. We shall first carry out the integration on  $x$ , for a fixed  $t$ . It will therefore be possible to choose the (complex) integration paths for  $x$  dependent on  $t$ . The integration paths are modified as follows:  $\tau_0$  is replaced by  $\tau$ , the semi-circle with radius unity lying in the domain  $\mathcal{I}_m(t) \geq 0$ . Therefore,  $\arg(t)$  varies from  $\pi$  to 0 when  $t$  moves from  $-1$  to  $+1$ . On the other hand,  $\zeta_0$  is replaced by  $\zeta_{\psi_t}$ , the line starting from the origin and forming an angle

$$\psi_t = \frac{\pi}{2} - \arg(t) \quad (194)$$

with the real axis. With these new contours,

$$R_{\parallel}(\lambda) = \frac{1}{\pi} \text{Re} \int_{\tau} \frac{t^2 dt}{\sqrt{1-t^2}} \int_{\zeta_{\psi_t}} e^{ixt} \frac{J_1(\lambda x)}{x} dx. \quad (195)$$

When  $t$  moves from  $-1$  to  $+1$ ,  $\psi_t$  varies from  $-\frac{\pi}{2}$  to  $+\frac{\pi}{2}$ . The argument of  $ixt$  is equal to  $\pi$ . Therefore, the real part of  $ixt$  is always negative and the function  $e^{ixt}$  decays exponentially to zero as  $|x| \rightarrow \infty$ . Therefore, with the new paths of integration  $\tau$  and  $\zeta_{\psi_t}$ , it is possible to expand the integrand of equation (195) in power series of  $\lambda$ , for  $\lambda \rightarrow 0$ , and integrate term by term:

$$R_{\parallel}(\lambda) = \frac{1}{\pi} \text{Re} \int_{\tau} \frac{t^2 dt}{\sqrt{1-t^2}} \int_{\zeta_{\psi_t}} e^{ixt} \left( \frac{\lambda}{2} - \frac{\lambda^3 x^2}{16} + \dots \right) dx. \quad (196)$$

Setting  $ixt = -y$  ( $y$  real  $\geq 0$ ), we get

$$R_{\parallel}(\lambda) = -\frac{1}{\pi} \text{Re} \int_{-1}^{+1} \frac{t^2 dt}{\sqrt{1-t^2}} \times \int_0^{+\infty} e^{-y} \left( \frac{\lambda}{2} + \frac{\lambda^3 y^2}{16t^2} + \dots \right) \frac{dy}{it}, \quad (197)$$

where we recall that  $t$  is a complex variable and the integration has to be performed over the semi-circle of radius unity lying on the domain  $\mathcal{I}_m(t) \geq 0$ . Writing  $t = e^{i\theta}$ , we find that

$$\text{Re} i \int_{-1}^{+1} \frac{t dt}{\sqrt{1-t^2}} = 0, \quad (198)$$

and

$$\text{Re} i \int_{-1}^{+1} \frac{dt}{t\sqrt{1-t^2}} = \pi. \quad (199)$$

Therefore,

$$R_{\parallel}(\lambda) \sim \frac{\lambda^3}{16} \Gamma(3) = \frac{\lambda^3}{8}. \quad (200)$$

Using a similar procedure, we find that

$$R_{\perp}(\lambda) \sim \frac{\lambda}{2}. \quad (201)$$

Note that this second result can also be obtained directly from equation (128) by using  $J_1(\lambda x)/x \sim \lambda/2$  and

$$\int_0^{+\infty} \frac{J_1(x)}{x} dx = 1. \quad (202)$$

### B.2 Asymptotic behaviors for $\lambda \rightarrow +\infty$

We now determine the asymptotic behaviors of the functions (127) and (128) for  $\lambda \rightarrow +\infty$ . Setting  $z = \lambda x$ , we can rewrite equation (127) as

$$R_{\parallel}(\lambda) = \int_0^{+\infty} dz \frac{J_1(z)}{z} \left[ J_0 \left( \frac{z}{\lambda} \right) - \frac{J_1 \left( \frac{z}{\lambda} \right)}{\frac{z}{\lambda}} \right]. \quad (203)$$

Using identity (114), we obtain

$$R_{\parallel}(\lambda) = \frac{1}{\pi} \int_0^{\pi} d\theta \sin^2 \theta \int_0^{+\infty} dz e^{iz \cos \theta} \times \left[ J_0 \left( \frac{z}{\lambda} \right) - \frac{J_1 \left( \frac{z}{\lambda} \right)}{\frac{z}{\lambda}} \right]. \quad (204)$$

Setting  $t = \cos \theta$ , using the contours introduced in the previous section and expanding the last term in Taylor series for  $1/\lambda \rightarrow 0$ , we find that

$$R_{\parallel}(\lambda) = \frac{1}{\pi} \text{Re} \int_{\tau} dt \sqrt{1-t^2} \int_{\zeta_{\psi_t}} e^{izt} \left[ \frac{1}{2} - \frac{3}{16} \frac{z^2}{\lambda^2} + \dots \right] dz. \quad (205)$$

Setting  $izt = -y$  ( $y$  real  $\geq 0$ ) we get

$$R_{\parallel}(\lambda) = -\frac{1}{\pi} \operatorname{Re} \int_{-1}^{+1} dt \sqrt{1-t^2} \times \int_0^{+\infty} e^{-y} \left[ \frac{1}{2} + \frac{3}{16} \frac{y^2}{\lambda^2 t^2} + \dots \right] \frac{dy}{it}, \quad (206)$$

where the integration on  $t$  has to be carried out on the upper semi-circle in the complex plane. Using the identities

$$\operatorname{Re} i \int_{-1}^{+1} \sqrt{1-t^2} \frac{dt}{t} = \pi, \quad (207)$$

$$\operatorname{Re} i \int_{-1}^{+1} \sqrt{1-t^2} \frac{dt}{t^3} = -\frac{\pi}{2}, \quad (208)$$

we find that

$$R_{\parallel}(\lambda) = \frac{1}{2} - \frac{3}{16\lambda^2} + \dots \quad (209)$$

Using a similar procedure, we obtain

$$R_{\perp}(\lambda) = \frac{1}{2} - \frac{1}{16\lambda^2} + \dots \quad (210)$$

### B.3 Relation to hypergeometric functions and behavior for $\lambda \sim 1$

In fact, the functions (127) and (128) can be expressed in terms of Hypergeometric functions. We can then easily obtain their asymptotic behaviors from standard formulae [45]. We have

$$R_{\parallel}(\lambda) = \frac{\lambda}{2} \left[ F\left(\frac{1}{2}, \frac{1}{2}, 2, \lambda^2\right) - F\left(-\frac{1}{2}, \frac{1}{2}, 2, \lambda^2\right) \right], \quad (211)$$

$$R_{\perp}(\lambda) = \frac{\lambda}{2} F\left(-\frac{1}{2}, \frac{1}{2}, 2, \lambda^2\right). \quad (212)$$

For  $\lambda \rightarrow 0$ ,

$$R_{\parallel}(\lambda) = \frac{\lambda^3}{8} + \frac{\lambda^5}{32} + \frac{15\lambda^7}{1024} + \dots \quad (213)$$

$$R_{\perp}(\lambda) = \frac{\lambda}{2} - \frac{\lambda^3}{16} - \frac{\lambda^5}{128} - \frac{5\lambda^7}{2048} + \dots \quad (214)$$

and for  $\lambda \rightarrow +\infty$ ,

$$R_{\parallel}(\lambda) = \frac{1}{2} - \frac{3}{16\lambda^2} - \frac{5}{128\lambda^4} + \dots \quad (215)$$

$$R_{\perp}(\lambda) = \frac{1}{2} - \frac{1}{16\lambda^2} - \frac{1}{128\lambda^4} + \dots \quad (216)$$

We also note the particular values

$$R_{\parallel}(1) = \frac{2}{3\pi}, \quad R_{\perp}(1) = \frac{4}{3\pi}. \quad (217)$$

From equation (211), we get

$$R'_{\parallel}(\lambda) = \frac{1}{2} \left[ F\left(\frac{1}{2}, \frac{1}{2}, 2, \lambda^2\right) - F\left(-\frac{1}{2}, \frac{1}{2}, 2, \lambda^2\right) \right] + \frac{\lambda^2}{8} \left[ F\left(\frac{1}{2}, \frac{3}{2}, 3, \lambda^2\right) + F\left(\frac{3}{2}, \frac{3}{2}, 3, \lambda^2\right) \right]. \quad (218)$$

This function diverges for  $\lambda \rightarrow 1$  like

$$R'_{\parallel}(\lambda) = \frac{4}{3\pi} - \frac{1}{\pi} (2\gamma + \ln 2 + \ln |1-\lambda| + 2\psi(3/2)), \quad (219)$$

where  $\gamma = 0.577216\dots$  is the Euler constant and  $\psi(3/2) = 0.03649\dots$  is the Digamma function [45].

## Appendix C: Regularization of the linear divergence

In this Appendix, we show how the linear divergence of the diffusion coefficient for a Coulombian plasma in  $d = 2$  can be regularized by taking into account collective effects. When collective effects are taken into account using equation (7) instead of equation (8), the diffusion coefficient is given by

$$D^{\mu\nu} = \pi(2\pi)^d m \int d\mathbf{v}_1 d\mathbf{k} k^{\mu} k^{\nu} \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2} \delta(\mathbf{k} \cdot \mathbf{u}) f(\mathbf{v}_1). \quad (220)$$

We concentrate here on a thermal bath with Maxwellian distribution. Using the same method as in Section 2.5 but keeping the dielectric function, we obtain

$$D^{\mu\nu} = \pi(2\pi)^d m \rho \left( \frac{\beta m}{2\pi} \right)^{1/2} \times \int d\mathbf{k} \frac{k^{\mu} k^{\nu}}{k} \frac{\hat{u}(k)^2}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2} e^{-\beta m \frac{(\mathbf{k} \cdot \mathbf{v})^2}{2k^2}}. \quad (221)$$

For a thermal bath one has (see, e.g., [22]):

$$|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2 = (1 - \eta(k) B(\hat{\mathbf{k}} \cdot \mathbf{x}))^2 + C(\hat{\mathbf{k}} \cdot \mathbf{x})^2, \quad (222)$$

where we have defined  $\hat{\mathbf{k}} = \mathbf{k}/k$ ,  $\mathbf{x} = (\beta m/2)^{1/2} \mathbf{v}$ ,  $\eta(k) = -(2\pi)^d \hat{u}(k) \beta m \rho$ ,  $B(z) = 1 - 2ze^{-z^2} \int_0^z e^{t^2} dt$  and  $C(z) = \sqrt{\pi} |z| e^{-z^2}$ . The diffusion tensor can be rewritten

$$D^{\mu\nu} = \frac{\pi}{(2\pi)^d \beta^2 \rho m} \left( \frac{\beta m}{2\pi} \right)^{1/2} \int d\hat{\mathbf{k}} \hat{k}^{\mu} \hat{k}^{\nu} I(\hat{\mathbf{k}} \cdot \mathbf{x}) e^{-(\hat{\mathbf{k}} \cdot \mathbf{x})^2}, \quad (223)$$

where

$$I(z) = \int_0^{+\infty} \frac{k^d \eta(k)^2 dk}{(1 - \eta(k) B(z))^2 + C(z)^2}. \quad (224)$$

For a Coulombian potential, one has  $\eta(k) = -k_D^2/k^2$  and we obtain

$$I(z) = \int_0^{+\infty} \frac{k^d dk}{(B(z) + k^2/k_D^2)^2 + C(z)^2}. \quad (225)$$

This can be rewritten  $I(z) = k_D^{d+1} C(z)^{\frac{d-3}{2}} \Phi_d(B(z)/C(z))$  where  $\Phi_d(z) = \int_0^{+\infty} t^d dt / ((z + t^2)^2 + 1)$ . If we neglect collective effects we have instead

$$I^{Landau} = k_D^4 \int_0^{+\infty} \frac{dk}{k^{4-d}}, \quad (226)$$

which presents divergences for  $k \rightarrow 0$  if  $d \leq 3$ . The regularization of the divergence in  $d = 3$  has been treated by Balescu [8] and the regularization in  $d = 1$  has been treated in [22]. Let us focus here on the case  $d = 2$ . We note that, contrary to equation (226), the integral (225) is well-behaved for  $k \rightarrow 0$  and  $k \rightarrow +\infty$ . Therefore, there is no linear divergence of the diffusion coefficient and friction force when collective effects are taken into account.

To obtain the expression of the diffusion tensor, one has to substitute equation (225) in equation (223), using  $z = x \cos \theta$ , and carry out the integrations. We shall simplify the calculations a little bit by introducing an approximate analytical expression of  $I(z)$  (we have checked numerically that the exact treatment yields close results). Let us first consider asymptotic behaviors of the previously defined functions. We have  $\Phi_2(z) = (\pi/(2\sqrt{2}))(1 - z/2 + z^2/8 + \dots)$  for  $z \rightarrow 0$ ,  $\Phi_2(z) \sim \pi/(4\sqrt{z})$  for  $z \rightarrow +\infty$  and  $\Phi_2(z) \sim (\pi/2)\sqrt{-z}$  for  $z \rightarrow -\infty$ . On the other hand,  $B(z)/C(z) \sim 1/(\sqrt{\pi}|z|)$  for  $z \rightarrow 0$  and  $B(z)/C(z) \sim -1/(2\sqrt{\pi}|z|^3)e^{z^2}$  for  $|z| \rightarrow +\infty$ . Thus, we find that  $I(z)/k_D^3 \rightarrow \pi/4$  for  $z \rightarrow 0$  and  $I(z)/k_D^3 \rightarrow \sqrt{\pi}/(2\sqrt{2}z^2)e^{z^2}$  for  $z \rightarrow +\infty$ . Let us consider a simple interpolation formula of the form

$$\frac{I(z)}{k_D^3} = \frac{\pi}{4} + \frac{1}{2} \left(\frac{\pi}{2}\right)^{1/2} \left(\frac{e^{z^2}}{z^2} - \frac{1}{z^2} - 1\right). \quad (227)$$

With this expression, it turns out that the trace of the diffusion tensor

$$D^{\mu\mu} = D_0 \int_0^{2\pi} I(x \cos \theta) e^{-x^2 \cos^2 \theta} d\theta, \quad (228)$$

can be calculated analytically ( $D_0$  is the value of the constant in front of the integral in Eq. (223)). When we use equation (227), we obtain

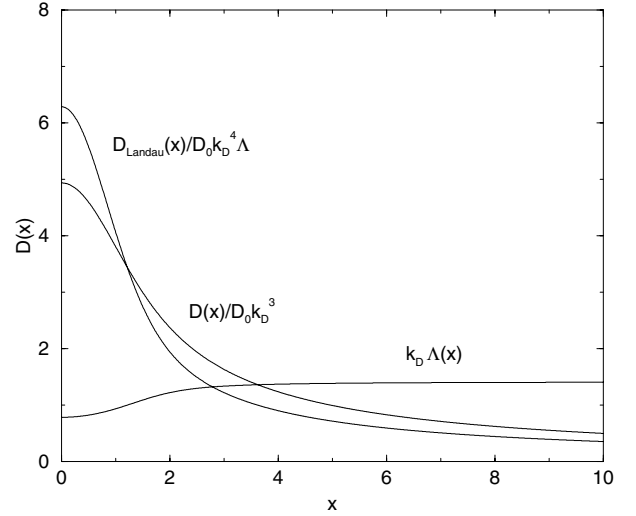
$$D^{\mu\mu} = D_0 k_D^3 \frac{\pi^{3/2}}{2} e^{-\frac{x^2}{2}} \left[ \sqrt{\pi} I_0 \left(\frac{x^2}{2}\right) + \sqrt{2} I_1 \left(\frac{x^2}{2}\right) \right]. \quad (229)$$

Alternatively, when we use equation (226), we get

$$D_{Landau}^{\mu\mu} = D_0 k_D^4 2\pi \Lambda e^{-\frac{x^2}{2}} I_0 \left(\frac{x^2}{2}\right), \quad (230)$$

where  $\Lambda = \int_0^{+\infty} dk/k^2$ . We note that the trace of the diffusion coefficient decays like  $x^{-1}$  in each case. We can write equation (229) in the form of equation (230):

$$D^{\mu\mu} = D_0 k_D^4 2\pi \Lambda(x) e^{-\frac{x^2}{2}} I_0 \left(\frac{x^2}{2}\right), \quad (231)$$



**Fig. C.1.** Dependence of the trace of the diffusion tensor with the velocity, using (i) the Landau approximation or (ii) the Lenard-Balescu treatment of collective effects.

where  $\Lambda(x)$  is now a function of  $x$  which is perfectly well defined (without divergence). Using equation (229) and comparing with equation (231) we find for  $x \rightarrow 0$  that

$$\Lambda(0) = \frac{\pi}{4k_D} \simeq \frac{0.785\dots}{k_D}, \quad (232)$$

and for  $x \rightarrow +\infty$  that

$$\Lambda(+\infty) = \frac{\pi + \sqrt{2\pi}}{4k_D} \simeq \frac{1.412\dots}{k_D}. \quad (233)$$

This justifies, without introducing an ad hoc large-scale cut-off, that  $\Lambda$  is of order the Debye length  $k_D^{-1}$  as expected. We also note that the function

$$k_D \Lambda(x) = \frac{1}{4} \left( \pi + \sqrt{2\pi} \frac{I_1(x^2/2)}{I_0(x^2/2)} \right), \quad (234)$$

does not vary crucially with  $x$  and remains of order unity (see Fig. C.1). Therefore, using the Landau approximation and introducing a large scale cut-off at the Debye length  $k_D^{-1}$  seems to be a reasonably good approximation. However, when one evaluates the component  $D_{\parallel}(x)$  of the diffusion coefficient, one finds that the treatment using equation (225) leads to a decay like  $x^{-2}$  for  $x \rightarrow +\infty$  while the Landau approximation (226) leads to a decay like  $x^{-3}$  (see Sect. 4.2). Therefore, in that case, there are qualitative discrepancies between the two approaches.

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